COMPUTABILITY (25/11/2024)

EXERCISE : URM^P

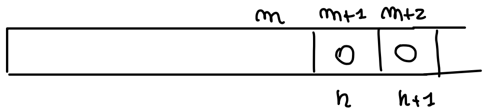
$Z(n)$
$T(m, n)$
$J(m, n, t)$
~~$S(n)$~~ $P(n)$

$z_n \leftarrow z_n \dot{-} 1 = \begin{cases} 0 & \text{if } z_n = 0 \\ z_n - 1 & \text{if } z_n > 0 \end{cases}$

$\mathcal{C}^P \subseteq \mathcal{C}$?

$(\mathcal{C}^P \subseteq \mathcal{C})$

given a program of URM^P machine P



P

t $P(n)$ $J(1,1, SUB)$

$m = \rho(P)$

SUB : $J(n, m+1, END)$
$S(m+2)$
LOOP : $J(n, m+2, RES)$
$S(m+1)$
$S(m+2)$
$J(1,1, LOOP)$
RES: $T(m+1, n)$
END: $J(1,1,t+1)$

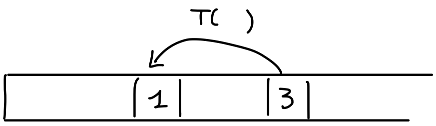
FORMAL PROOF:

For every program P ~~of~~ ~~URM^P~~, $k \in \mathbb{N}$ there is a program P' of URM using $Z(n)$, $T(m,n)$, $J(m,n,t)$, $S(n)$, $P(n)$

s.t. $f_P^{(k)} = f_{P'}^{(k)}$

by induction on the number of instructions $P(n)$

$(\mathcal{C}^P \subsetneq \mathcal{C})$



Given a program P and $x \in \mathbb{N}^k$ the maximum value in the register after any number of steps of $P(\vec{x})$ is always bounded by $\max_{1 \le i \le k} x_i$

Proof by induction the number t of steps of $P(\vec{x})$

$(t=0)$ the memory is

1		K			
x_1	----	x_K	0	0	---

$$\max_i r_i = \max_{1 \le i \le K} x_i$$

$(t \to t+1)$ the content of the memory after $t+1$ steps is

x_1	---	x_K

t ⇝

r_1'	----	r_K'	---	r_m'

$m = \rho(P)$

by inductive hyp.

$$\max_i r_i' \le \max_{1 \le i \le K} x_i$$

$t+1$ ⇝

r_1	---	r_K	---	r_m

last step can be

$Z(n)$

$T(m,n)$

$J(m,n,t)$

$P(n)$

$$\max_i r_i \le \max_i r_i' \le \max_{1 \le i \le K} x_i$$

* The successor function $S: \mathbb{N} \to \mathbb{N}$ $S(x) = x+1$ $S \notin \mathcal{C}^P$

0	0	----

$S(0) = 1$

~~⇝~~

1	

← This would increase the maximum value in the memory. Impossible.

<u>NOTE</u>: Termination is decidable in URM^P

$x_1 \ldots\ldots x_K$	0	0	----

⇝

1		m
r_1	----	r_m

$\subseteq [0,h]^m$ finite

$m = \rho(P)$

$$r_j \le \max_{1 \le i \le K} x_i = h$$

EXERCISE: Show that there is a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that

$$E_{k(x)} = W_x$$

$P_x \rightsquigarrow P_{k(x)}$ with outputs = inputs where P_x halts

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def P_k(x) (y) :
    P_x (y)
    return y
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define $f : \mathbb{N}^2 \to \mathbb{N}$

$$f(x, y) = \begin{cases} y & \text{if } \varphi_x(y)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}(\underbrace{\Psi_U(x,y)}_{= \varphi_x(y)}) \cdot y$$

where $\mathbb{1}(\varphi_x(y)) = \begin{cases} 1 & \text{if } \varphi_x(y)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$

computable (by composition)

By smn there exists $k : \mathbb{N} \to \mathbb{N}$ total computable such that $\forall x, y$

$$\varphi_{k(x)}(y) = f(x,y) = \begin{cases} y & \text{if } \varphi_x(y)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

* k is the desired function $\boxed{W_x = E_{k(x)}}$

($W_x \subseteq E_{k(x)}$) let $y \in W_x$ i.e. $\varphi_x(y)\downarrow$. Then $\varphi_{k(x)}(y) = f(x,y) = y$

hence $y \in E_{k(x)}$

($E_{k(x)} \subseteq W_x$) let $y \in E_{k(x)}$ i.e. $\exists z \in \mathbb{N}$ s.t. $\varphi_{k(x)}(z) = y$, where $\varphi_{k(x)}(z) = f(x,z)$

by definition of f, necessarily $z = y$ and $\varphi_x(y)\downarrow$

hence $y \in W_x$. $\square$

EXERCISE: there is a total computable function $K : \mathbb{N} \to \mathbb{N}$ s.t.

$$W_{K(x)} = \mathbb{P} \quad \text{(even numbers)}$$

$$E_{K(x)} = \{ z \in \mathbb{N} \mid z \geq x \} = [x, \infty)$$

Define $f : \mathbb{N}^2 \to \mathbb{N}$

$$f(x, y) = \begin{cases} x + y/2 & \text{if } y \text{ even} \\ \uparrow & \text{otherwise} \end{cases}$$

$$= (x + qt(2, y)) \cdot (1 + \mu z. \, rm(2, y))$$

$\bar{sg}(\mu z. \, rm(2,y))$

$rm(2, y)$: 0 if y even, 1 otherwise

$\mu z. \, rm(2,y)$: 0 if y is even, $\uparrow$ otherwise

$1 + \mu z. \, rm(2,y)$: 1 if y even, $\uparrow$ otherwise

computable

Hence by the smn theorem there is $K : \mathbb{N} \to \mathbb{N}$ total computable function such that $\forall x, y$

$$\varphi_{K(x)}(y) = f(x, y) = \begin{cases} x + y/2 & \text{if } y \text{ even} \\ \uparrow & \text{otherwise} \end{cases}$$

$*$ K is the desired function

$\to$ $W_{K(x)} = \mathbb{P}$ ok, by construction

$\to$ $E_{K(x)} = \{ z \mid z \geq x \}$

$$E_{K(x)} = \{ \varphi_{K(x)}(y) \mid y \in \underbrace{W_{K(x)}}_{\mathbb{P}} \}$$

$$= \{ \varphi_{K(x)}(2y) \mid y \in \mathbb{N} \}$$

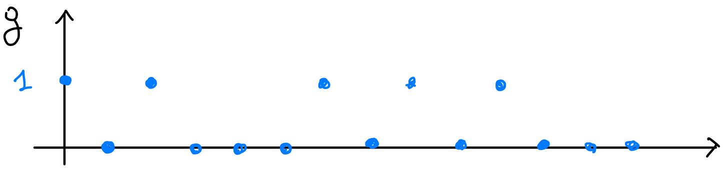
$$= \{ x + \frac{2y}{2} \mid y \in \mathbb{N} \}$$

$$= \{ x + y \mid y \in \mathbb{N} \}$$

$$= \{ z \mid z \geq x \}$$

EXERCISE: Are there functions $f, g : \mathbb{N} \to \mathbb{N}$ f computable, g not computable

s.t. $f \circ g$ computable ?



$$g(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

non computable

$f(x) = 0 \quad \forall x$ computable

$$f(g(x)) = f(x) = 0 \quad \forall x$$

computable

* Are there $f, g : \mathbb{N} \to \mathbb{N}$ f, g non computable s.t. $f \circ g$ computable?

YES

$$g(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases} \qquad \text{non computable}$$

$$f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ \varphi_x(x)+1 & \text{if } x > 1 \text{ and } \varphi_x(x)\downarrow \\ 0 & \text{if } x > 1 \text{ and } \varphi_x(x)\uparrow \end{cases}$$

f not computable

- $\forall x > 1 \quad f \neq \varphi_x$ since $f(x) \neq \varphi_x(x)$
- $\forall y \in \mathbb{N} \quad \exists x > 1 \quad \varphi_y = \varphi_x$ hence $f \neq \varphi_x = \varphi_y \Rightarrow f \neq \varphi_y$

but $f \circ g(x) = f(g(x)) = 0 \quad \forall x$

EXERCISE: Every computable function $f : \mathbb{N} \to \mathbb{N}$ can be expressed as the composition of two non-computable functions.

<u>EXERCISE</u> : Prove that

$$pow_2 : \mathbb{N} \to \mathbb{N}$$
$$pow_2(x) = 2^x \qquad \text{is in } \mathcal{PR}$$

by using only the definition.

(least class of functions including the basic functions i.e. zero, successor and projections, and closed by composition and primitive recursion)

$x + y$ $\qquad x + 0 = x$
$\qquad\qquad x + (y+1) = (x+y) + 1$

$x * y$ $\qquad x * 0 = 0$
$\qquad\qquad x * (y+1) = (x*y) + x$

x^y $\qquad x^0 = 1$
$\qquad\qquad x^{y+1} = (x^y) * x$

$$pow2(x) = 2^x = succ(succ(0))^x$$

alternatively

$$\begin{cases} pow_2(0) = 2^0 = 1 = succ(0) \\ pow_2(y+1) = 2^{y+1} = 2^y + 2^y = pow_2(y) + pow_2(y) \end{cases}$$

and observe that $+$ is in $\mathcal{PR}$

even more direct

$$\begin{cases} pow_2(0) = 2^0 = 1 = succ(0) \\ pow_2(y+1) = 2^{y+1} = 2 \cdot 2^y = twice(pow_2(y)) \end{cases}$$

$$\begin{cases} twice(0) = 0 \\ twice(y+1) = 2(y+1) = 2y + 2 = twice(y) + 2 = succ(succ(twice(y))) \end{cases}$$

EXERCISE : show that $\chi_{\mathbb{P}} : \mathbb{N} \to \mathbb{N}$

$$\chi_{\mathbb{P}}(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$ is in PR

using only the definition of PR.

$$\begin{cases} \chi_{\mathbb{P}}(0) = 1 \\ \chi_{\mathbb{P}}(y+1) = \overline{sg}(\chi_{\mathbb{P}}(y)) \end{cases}$$

$$\begin{cases} \overline{sg}(0) = 1 = succ(0) \\ \overline{sg}(y+1) = 0 \end{cases}$$