

Teorema di Taylor

$x_0 \in \mathbb{R}$, f che sia derivabile infinite volte
in un intervallo che contiene il pto x_0
(cioè in un intervallo del tipo $(x_0 - r, x_0 + r)$)

Allora $\forall N \in \mathbb{N} \quad \exists P_{N, x_0}(x)$ di grado $\leq N$

tale che $\lim_{x \rightarrow x_0} \frac{f(x) - P_{N, x_0}(x)}{|x - x_0|^N} = 0$

$$(f(x) = P_{N, x_0}(x) + \theta(|x - x_0|^N)) \rightarrow \begin{aligned} \theta((x-x_0)^k) &= \\ \lim_{x \rightarrow x_0} \frac{\theta(x)}{(x-x_0)^k} &= 0 \end{aligned}$$

$$\begin{aligned}
 P_{N, x_0}(x) &= \underbrace{f(x_0)} + f'(x_0) \cdot (x - x_0) + \\
 &+ \frac{1}{2} f''(x_0) (x - x_0)^2 + \frac{1}{3!} f^{(3)}(x_0) (x - x_0)^3 + \\
 &+ \frac{1}{4!} f^{(4)}(x_0) (x - x_0)^4 + \dots + \frac{1}{N!} f^{(N)}(x_0) (x - x_0)^N.
 \end{aligned}$$

Il polinomio di Taylor a volte ci permette di capire come è fatta una funzione ^{continua a x₀} quando non riesce a studiare il segno delle derivate.

$$f(x) = e^x - \frac{1}{1-x}$$

$$\begin{aligned} D &= \{x \neq 1\} \\ &= (-\infty, 1) \cup (1, +\infty) \end{aligned}$$

$$\lim_{x \rightarrow 1^+} e^x - \frac{1}{1-x} = e^1 - \frac{1}{0^-} = e - (-\infty) = +\infty$$

$\lim_{x \rightarrow 1^-} e^x - \frac{1}{1-x} = e^1 - \frac{1}{0^+} = -\infty$

$x=1$ ASINT VERTICALE

$$f(0) = 0$$

$$f(x) = e^x - \frac{1}{1-x} = e^x - [1-x]^{-1}$$

$$f'(x) = e^x - \left[(-1)(1-x)^{-1-1} \cdot (0-1) \right]$$

$$= (x)^{\alpha-1} = \underline{\underline{\alpha x^{\alpha-1}}}$$

derivative
d; $-1-x$

$$f'(x) = e^x - (1-x)^{-2}$$

\checkmark \checkmark

$f \bar{e}$
derivable $\forall x \neq 1$

$$f'(0) = 1 - (1)^{-2} = 0$$

voglio capire come è fatto f vicino a $x=0$.

$$f(x) = e^x - (1-x)^{-1}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$x \rightarrow 0$

$\alpha \in \mathbb{R}$

$$(1+x)^\alpha = 1 + \underline{\underline{\alpha}} \cdot \underline{\underline{x}} + \frac{1}{2} \underline{\underline{\alpha}} \cdot (\underline{\underline{\alpha}}-1) \underline{\underline{x^2}} + o(x^2)$$

$$(1-x)^{-1} = [1 + (-x)]^{-1} = 1 + \underline{(-1)} \underline{(-x)} + \\ + \frac{1}{2} \underline{(-1)} \underline{(-1-1)} \underline{(-x)^2} + o(-x)^2 = 1 + x +$$

$$+ \frac{1}{2} \cdot \underline{(-1)} \cdot \underline{(-2)} \underline{(x^2)} + o(x^2) = 1 + x + x^2 + o(x^2)$$

$$\begin{aligned}
 f(x) &= e^x - (1-x)^{-1} = \\
 &= 1 + x + \frac{1}{2}x^2 + \Theta(x^2) - \boxed{1 + x + x^2 + \Theta(x^3)} \\
 &= \cancel{1 + x + \frac{1}{2}x^2 + \Theta(x^2)} - \cancel{1 - x - x^2 + \Theta(x^3)} \\
 &= -\frac{1}{2}x^2 + \Theta(x^2) \quad \text{per } x \rightarrow 0 \\
 &= -\frac{1}{2}x^2 [1 + \Theta(1)]
 \end{aligned}$$

↪ $x=0$ pbo di MAX LOCALE



linee

$$x \rightarrow +\infty$$

$$e^x - \frac{1}{1-x} = +\infty + 0 = +\infty$$

+ ∞ non ito
AS. ORIZZONTALE

lim

$$x \rightarrow -\infty$$

$$e^x - \frac{1}{1-x} = 0 + 0 = 0$$

$e^{-\infty} = 0$

$\frac{1}{+\infty} = 0$

$$y = 0$$

ASINT ORIZZ

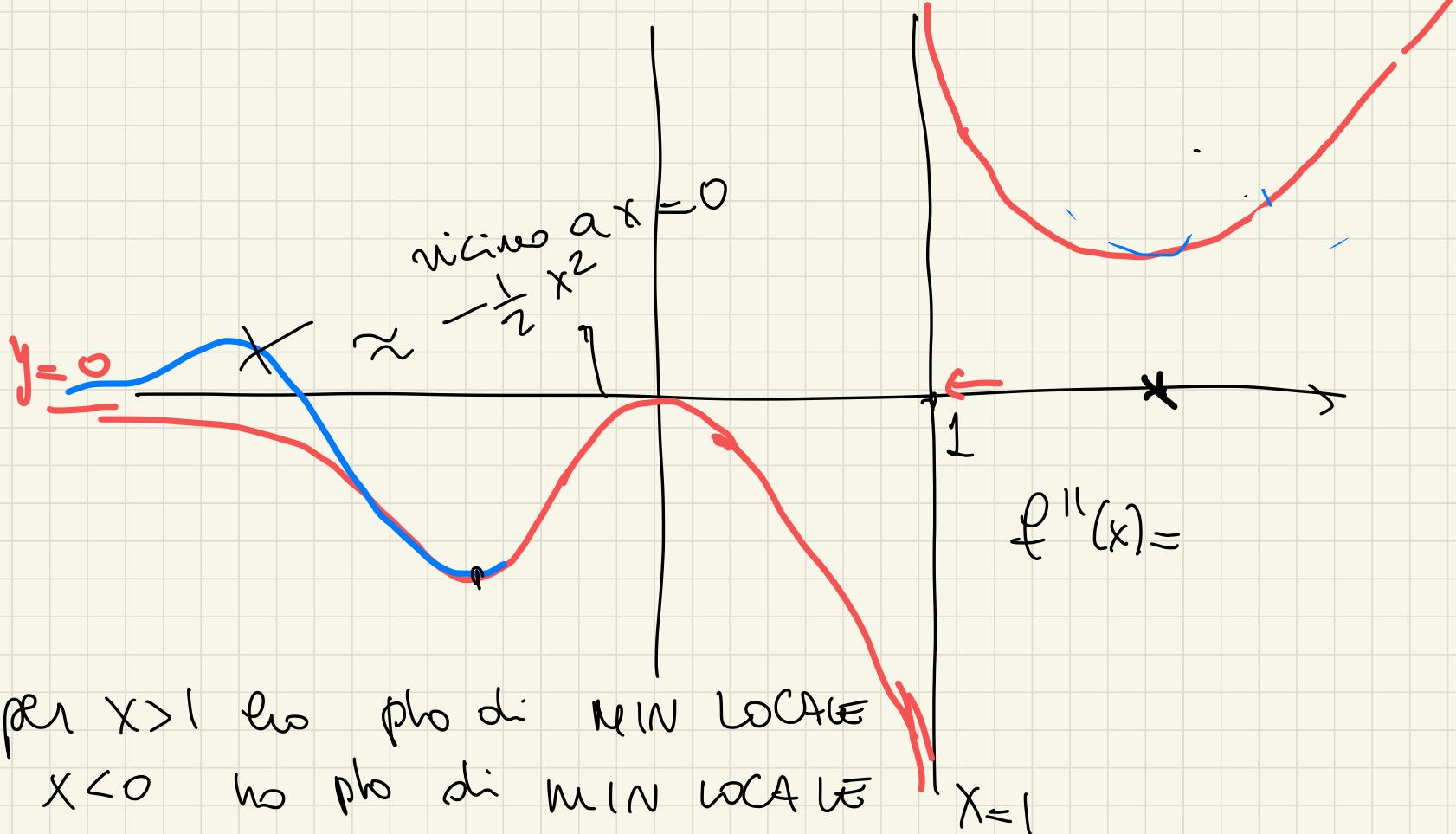
$$a - \infty$$

Caso análogo obligado a ∞

Líneal $\frac{f(x)}{x} = \lim_{x \rightarrow \infty}$ $e^x - \frac{1}{(1-x)}$ =

$$\lim_{x \rightarrow \infty} \frac{e^x \left[1 - \frac{1}{(1-x)} \right]}{x} = \infty$$

Otro caso análogo obligado a ∞



$$f'(x) = e^x - \frac{1}{(1-x)^2} = e^x - \underbrace{(1-x)^{-2}}_{\text{green shaded}}$$

$$f''(x) = e^x - \left[\underbrace{(-2)}_{\text{green bracket}} (1-x)^{-2-1} \cdot \underbrace{(0-1)}_{\text{green bracket}} \right] =$$

$$(x^\alpha)' = \alpha \cdot x^{\alpha-1}$$

$$= e^x - \underbrace{\frac{2}{(1-x)^3}}_{\text{blue bracket}}$$

$$x > 1 \quad (1-x) < 0$$

$$(1-x)^3 < 0$$

$$-\frac{2}{(1-x)^3} > 0$$

$\&$ $f''(x) > 0$

ES

Determineen al waarden die $\alpha > 0$

$$\lim_{x \rightarrow 0^+} \frac{x \cos x - \sin x}{e^{(\alpha)} - 1}$$

N

$$\cos x = 1 - \frac{1}{2}x^2 + O(x^2)$$

$$\sin x = x - \frac{1}{6}x^3 + O(x^3)$$

$$\begin{aligned} x \cos x - \sin x &= x \left[1 - \frac{1}{2}x^2 + O(x^2) \right] - \left(x - \frac{1}{6}x^3 + O(x^3) \right) \\ &= \cancel{x - \frac{1}{2}x^3 +} \quad \text{yellow circle } O(x^2) - \cancel{x + \frac{1}{6}x^3 +} \quad \text{blue bracket } O(x^3) = \end{aligned}$$

$$= \left(-\frac{1}{2} + \frac{1}{6} \right) x^3 + o(x^3) = -\frac{1}{3} x^3 + o(x^3)$$

D: $e^{x^\alpha} - 1$ $\underbrace{\qquad}_{\alpha > 0}$

$$= x^3 \left[-\frac{1}{3} + o(1) \right]$$

$x \rightarrow 0^+$
 $x^\alpha \rightarrow 0^+$

$$e^x = 1 + x + o(x)$$

$$e^{x^\alpha} = 1 + x^\alpha + o(x^\alpha)$$

$$e^{x^\alpha} - 1 = x^\alpha + o(x^\alpha) = x^\alpha [1 + o(1)]$$

$$\lim_{x \rightarrow 0^+} \frac{x^3 \left[-\frac{1}{3} + o(1) \right]}{x^\alpha \left[1 + o(1) \right]} =$$

$$= \begin{cases} \alpha = 3 & = -\frac{1}{3} \\ 3 > \alpha & = 0 \\ 3 < \alpha & = -\infty \end{cases}$$

Es determinare al variare di $\alpha > 0$

lim
 $x \rightarrow +\infty$

The diagram illustrates the limit of the function $\arctg\left(\frac{1}{x^2}\right)$ as $x \rightarrow +\infty$. A yellow oval contains a black 'X' with a circled '3' above it, indicating the value of the function as $x^2 \rightarrow +\infty$. A red arrow points from this oval to a red 'plus infinity' symbol ($+\infty$). Another red arrow points from the same oval to the expression $\arctg\left(\frac{1}{x^2}\right)$, which is highlighted with a blue cloud. A third red arrow points from the expression to the result $\arctg 0 = 0$, which is written in blue. To the right, a red circle contains the variable x^α , with a red arrow pointing down to a red 'plus infinity' symbol ($+\infty$).

$$\lim_{x \rightarrow +\infty} \arctg\left(\frac{1}{x^2}\right) = \arctg 0 = 0$$

$$x \rightarrow 0$$

arctg

$$\text{arctg } x = x - \frac{1}{3}x^3 + o(x^3)$$

arctg $\left(\frac{1}{x^2}\right)$

$$x \rightarrow +\infty$$

arctg $\left(\frac{1}{x^2}\right)$

$$x \rightarrow +\infty$$

$$= \frac{1}{1 - \frac{1}{3}\left(\frac{1}{x^2}\right)^3} + o\left(\left(\frac{1}{x^2}\right)^3\right)$$

$$= \frac{1}{x^2} - \frac{1}{3} \cdot \frac{1}{x^6} + o\left(\frac{1}{x^6}\right)$$

$$x^3 \cdot \arctg \frac{1}{x^2} =$$

$$= x^3 \cdot \left[\frac{1}{x^2} - \frac{1}{3} \frac{1}{x^6} + o\left(\frac{1}{x^6}\right) \right] =$$

im vece sonette con termini
che valgono a 0 raccolgo

quelli di GRADO MINIMO

$$= x^3 \cdot \frac{1}{x^2} \left[1 - \frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \right] =$$

$$= x \left[1 - \frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \right].$$

$$\lim_{x \rightarrow +\infty} x^3 \arctg \frac{1}{x^2} - x^2 =$$

$$= \lim_{x \rightarrow +\infty} x^1 \left[1 - \frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \right] - x^2$$

$+ \infty$ $1 + 0 + 0 = 1$

(quando ho scritto di tenere le
potenze a ∞ non colgo quello d'
grado MAGGIORI)

δ $\alpha > 1$

$$\lim_{x \rightarrow +\infty} x \left[1 - \frac{1}{3} \frac{1}{x^u} + o\left(\frac{1}{x^4}\right) \right] - x^\alpha =$$

$$= \lim_{x \rightarrow +\infty} x^\alpha \left[\left(1 - \frac{1}{3} \frac{1}{x^u} + o\left(\frac{1}{x^4}\right) \right) - 1 \right]$$

x^α
 \downarrow
 $+\infty$

$$= +\infty \cdot (-1) = -\infty .$$

Se $x < 1$

$$\lim_{x \rightarrow +\infty} x \left[1 - \frac{1}{3} \cdot \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \right] - x =$$

$$= \lim_{x \rightarrow +\infty} x \left[1 - \frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) - 1 \right] =$$

$$= \lim_{x \rightarrow +\infty} x \left[-\frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \right] = \lim_{x \rightarrow +\infty} x \cdot \frac{1}{x^4} \left[-\frac{1}{3} + o(1) \right]$$
$$= 0 \cdot \left(-\frac{1}{3}\right) = 0$$

δ $\alpha < 1$

$$\lim_{x \rightarrow +\infty} x \cdot \left[1 - \frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \right] - x^\alpha$$

$$= \lim_{x \rightarrow +\infty} x \left[1 - \frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) - \frac{x^\alpha}{x} \right]$$

$+ \infty$

0 0 0

$$= +\infty \cdot 1 = +\infty.$$

Es

Calcolare al variare di $\alpha > 0$

$$\lim_{n \rightarrow +\infty}$$

$$\frac{n^\alpha \cdot \left[\lg \left(1 + \frac{1}{2n^2} \right) - 1 + \cos \left(\frac{1}{n} \right) \right]}{\left[\sin \left(\frac{1}{n} \right) - \frac{1}{n} \right]}$$

$0 - 0$

$$\operatorname{seu} \left(\frac{1}{n} \right) - \frac{1}{n}$$

$x \rightarrow 0$

$$\operatorname{seu} x = x - \frac{1}{6} x^3 + o(x^3) \quad x \rightarrow 0$$

$$\operatorname{Res} \left(\frac{1}{n} \right) = \frac{1}{n} - \frac{1}{6} \left(\frac{1}{n} \right)^3 + o\left(\frac{1}{n} \right)^3 \quad n \rightarrow +\infty$$

$$\left(\operatorname{seu} \left(\frac{1}{n} \right) - \frac{1}{n} \right) = \cancel{\frac{1}{n}} - \frac{1}{6} \left(\frac{1}{n} \right)^3 + o\left(\frac{1}{n} \right)^3 - \cancel{\frac{1}{n}}$$

$$= \frac{1}{n^3} \left[-\frac{1}{6} + o(1) \right] = n^{-3} \left[-\frac{1}{6} + o(1) \right]$$

$$\left(\frac{1}{n} \right)^3 = \frac{1}{n^3}$$

$$\underbrace{\lg \left(1 + \frac{1}{2n^2} \right)}_{\substack{x \rightarrow 0 \\ \text{approx}}}-1+\cos \left(\frac{1}{n} \right)$$

$n \rightarrow \infty$

$x \rightarrow 0$

$$\cos(x) = 1 - \frac{1}{2}x^2 + o(x^2)$$

$$\cos\left(\frac{1}{n}\right) = 1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + o\left(\frac{1}{n}\right)^2$$

$x \rightarrow 0$

$$\lg(1+x) = x + o(x)$$

$$\lg\left(1 + \frac{1}{2n^2}\right) = \frac{1}{2n^2} + o\left(\frac{1}{2n^2}\right)$$

$$\cancel{\frac{1}{2n^2}} + o\left(\frac{1}{n^2}\right) - 1 + 1 - \cancel{\frac{1}{2}\left(\frac{1}{n^2}\right)} + o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n^2}\right)$$

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$$

$$\cos \frac{1}{n} = 1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + o\left(\frac{1}{n}\right)^2$$

$$\log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$

$$\log\left(1 + \frac{1}{2n^2}\right) = \frac{1}{2n^2} - \frac{1}{2}\left(\frac{1}{2n^2}\right)^2 + o\left(\frac{1}{2n^2}\right)^2 =$$

$$= \boxed{\frac{1}{2n^2} - \frac{1}{2} \cdot \left(\frac{1}{4n^4}\right) + o\left(\frac{1}{n^4}\right)}$$

$$\log\left(1 + \frac{1}{2n^2}\right) - 1 + \cos \frac{1}{n} = \cancel{\frac{1}{2n^2}} - \frac{1}{8}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) - \cancel{1+1}$$
$$- \cancel{\frac{1}{2}\frac{1}{n^2}} + o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n^2}\right)$$

$$\cos x = 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 + o(x^4)$$

$$\cos\left(\frac{1}{n}\right) = 1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + \frac{1}{24}\left(\frac{1}{n}\right)^4 + o\left(\frac{1}{n^4}\right)$$

$$\lg\left(1 + \frac{1}{2n^2}\right) - 1 + \cos\left(\frac{1}{n}\right) =$$

$$\begin{aligned} &= \cancel{\frac{1}{2n^2}} - \frac{1}{8} \cancel{\frac{1}{n^4}} + o\left(\frac{1}{n^4}\right) - 1 + \cancel{1 - \frac{1}{2} \cancel{\frac{1}{n^2}}} + \\ &\quad + \cancel{\frac{1}{24} \cancel{\frac{1}{n^4}}} + o\left(\frac{1}{n^4}\right) = \left(-\frac{1}{8} + \frac{1}{24}\right) \frac{1}{n^4} + o\left(\frac{1}{n^4}\right) \end{aligned}$$

$$\log\left(1 + \frac{1}{2n^2}\right) - 1 + \cos \frac{1}{n} =$$

$$= -\frac{1}{12} \frac{1}{n^4} + o\left(\frac{1}{n^4}\right) = \frac{1}{n^4} \left[-\frac{1}{12} + o(1) \right]$$

$$= n^{-4} \left[-\frac{1}{12} + o(1) \right]$$

$$\left(\frac{1}{n}\right)^4 = \frac{1}{n^4} = n^{-4}$$

$$\lim_{n \rightarrow +\infty} n^\alpha \left[\lg\left(1 + \frac{1}{2n^2}\right) - \left[+ \cos \frac{1}{n}\right] \right] =$$

$n^\alpha \frac{1}{n} - \frac{1}{n}$

$$= \lim_{n \rightarrow +\infty} \frac{n^\alpha (n^{-4}) \left(-\frac{1}{12} + o(1) \right)}{n^{-3} \left(-\frac{1}{6} + o(1) \right)} =$$

$\frac{\cancel{n^\alpha}}{\cancel{n^{-3}}} \frac{\cancel{(n^{-4})}}{\cancel{(n^{-3})}} \frac{\left(-\frac{1}{12} + o(1) \right)}{\left(-\frac{1}{6} + o(1) \right)}$

$\xrightarrow{\alpha < 1} \frac{-\frac{1}{12}}{-\frac{1}{6}} = \frac{1}{2}$

$\xrightarrow{\alpha > 1} +\infty$

$\xrightarrow{\alpha = 1} 0$

aldmeeee visto che
 $n^\alpha \cdot \left(\log\left(1 + \frac{1}{2n^2}\right) - 1 + \cos \frac{1}{n} \right)$
 $\sim n^\alpha \cdot \frac{1}{n} - \frac{1}{n}$

per $n \rightarrow +\infty$
 $= n^{\alpha-1} \cdot \left[\frac{-\frac{1}{12} + o(1)}{-\frac{1}{6} + o(1)} \right] \approx n^{\alpha-1} = \left(\frac{1}{n} \right)^{1-\alpha}$
 $\downarrow \frac{1}{2}$

Ej Det al variare di $\alpha > 0$

lim $n \rightarrow +\infty$ $(n^{\alpha} - \arctan n + \lg n)$

$$\cdot \left[\sqrt[3]{1 + \frac{1}{n}} - e^{\frac{1}{3n}} \right]$$

$$\begin{aligned} n^{\alpha} - \arctan n + \lg n &= n^{\alpha} \left[1 - \frac{\arctan n}{n^{\alpha}} + \frac{\lg n}{n^{\alpha}} \right] \\ &\xrightarrow{n \rightarrow +\infty} 1 + o(1) \end{aligned}$$

$$\sqrt[3]{1 + \frac{1}{n}} - e^{\frac{1}{3n}}$$

$$n \rightarrow \infty \quad \frac{1}{n} \rightarrow 0$$

$$\frac{1}{3n} \rightarrow 0$$

$$\begin{aligned} x \rightarrow 0 \quad (1+x)^{\frac{1}{3}} &= 1 + \frac{1}{3}x + \frac{1}{2} \cdot \frac{1}{3} \left(\frac{1}{3}-1 \right) x^2 + o(x^2) \\ &= 1 + \frac{1}{3}x + \frac{1}{2} \cdot \frac{1}{3} \cdot \left(-\frac{2}{3} \right) x^2 + o(x^2) = \end{aligned}$$

$$\begin{aligned} x \rightarrow 0 \quad &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + o(x^2) \\ (1+x)^\alpha &= 1 + \alpha x + \frac{1}{2} \alpha(\alpha-1) x^2 + o(x^2) \end{aligned}$$

$$\begin{aligned} \sqrt[3]{1 + \frac{1}{n}} &= \left(1 + \frac{1}{n} \right)^{\frac{1}{3}} = 1 + \frac{1}{3} \cdot \frac{1}{n} - \frac{1}{9} \left(\frac{1}{n} \right)^2 + o \left(\frac{1}{n} \right)^2 \end{aligned}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$\begin{aligned} e^{\frac{1}{3n}} &= 1 + \frac{1}{3n} + \frac{1}{2} \cdot \left(\frac{1}{3n}\right)^2 + o\left(\frac{1}{3n}\right)^2 = \\ &= 1 + \frac{1}{3} \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{9} \frac{1}{n^2} + o\left(\frac{1}{n^2}\right) = \\ &= 1 + \frac{1}{3} \frac{1}{n} + \frac{1}{18} \frac{1}{n^2} + o\left(\frac{1}{n^2}\right) \end{aligned}$$

$$\sqrt[3]{1 + \frac{1}{n}} - e^{\frac{1}{3n}} =$$

$$= 1 + \frac{1}{3} \frac{1}{n} - \frac{1}{9} \frac{1}{n^2} + O\left(\frac{1}{n^2}\right) - \left[1 + \frac{1}{3} \frac{1}{n} + \frac{1}{18} \frac{1}{n^2} + O\left(\frac{1}{n^2}\right) \right]$$

$$= 1 + \frac{1}{3} \frac{1}{n} - \frac{1}{9} \frac{1}{n^2} + O\left(\frac{1}{n^2}\right) - 1 - \frac{1}{3} \frac{1}{n} - \frac{1}{18} \frac{1}{n^2} + O\left(\frac{1}{n^2}\right)$$

$$= \left(-\frac{1}{9} - \frac{1}{18} \right) \frac{1}{n^2} + O\left(\frac{1}{n^2}\right) = -\frac{1}{6} \frac{1}{n^2} + O\left(\frac{1}{n^2}\right) =$$

$$= \frac{1}{n^2} \left(-\frac{1}{6} + O(1) \right)$$

$$\lim_{n \rightarrow +\infty} \overbrace{M^\alpha \cdot (1+\mathcal{O}(1))}^{\text{yellow}} \cdot \underbrace{\frac{1}{n^2} \left(-\frac{1}{6} + \mathcal{O}(1) \right)}_{\text{yellow}} =$$

$$= \lim_{n \rightarrow +\infty} \frac{n^\alpha}{n^2} \underbrace{(1+\mathcal{O}(1))}_{\text{yellow}} \underbrace{\left(-\frac{1}{6} + \mathcal{O}(1) \right)}_{\text{yellow}} =$$

$$= \begin{cases} \alpha = 2 & = -\frac{1}{6} \\ \alpha > 2 & = (+\infty) \cdot 1 \cdot \left(-\frac{1}{6} \right) = -\infty \\ \alpha < 2 & = 0 \end{cases}$$