Teorema di Taylor

$x_0 \in \mathbb{R}$, f che sia derivabile infinite volte in un intervallo che contiene il pto x_0 (cioè in un intervallo del tipo (x_0-r, x_0+r))

Allora $\forall N \in \mathbb{N}$ $\exists\, P_{N,x_0}(x)$ di grado $\leq N$

tale che $\displaystyle\lim_{x \to x_0} \frac{f(x) - P_{N,x_0}(x)}{|x-x_0|^N} = 0$

$$\Big(f(x) = P_{N,x_0}(x) + o(|x-x_0|^N) \quad x \to x_0$$

$$o((x-x_0)^k) = \Big\{ h \;\; \lim_{x \to x_0} \frac{h(x)}{|x-x_0|^k} = 0 \Big\}$$

$$P_{N,x_0}(x) = \underbrace{f(x_0)} + f'(x_0)\cdot(x-x_0) +$$

$$+ \frac{1}{2} f''(x_0)\,(x-x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)\,(x-x_0)^3 +$$

$$+ \frac{1}{4!} f^{(4)}(x_0)\,(x-x_0)^4 + \dots + \frac{1}{N!} f^{(N)}(x_0)\,(x-x_0)^N .$$

Il polinomio di Taylor a volte mi permette di capire come è fatta una funzione vicino a x_0 quando non riesco a studiare il segno della derivata.

$$f(x) = e^x - \frac{1}{1-x}$$

$D = \{x \neq 1\}$
$= (-\infty, 1) \cup (1, +\infty)$

$$\lim_{x \to 1^+} e^x - \left[\frac{1}{1-x}\right] = e^1 - \frac{1}{0^-} = e - (-\infty) = +\infty$$

$e^x \to e^1$

$1-x \to 0^-$

$$\lim_{x \to 1^-} e^x - 1/x-1 = -\infty$$

$x = 1$ ASINT VERTICALE

$f(0) = 0$

$$f(x) = e^x - \frac{1}{1-x} = \underbrace{e^x - [1-x]^{-1}}$$

$$f'(x) = e^x - \left[(-1)(1-x)^{-1-1} \cdot \underbrace{(0-1)}_{\text{derivata di } 1-x}\right] =$$

$$(x^{\alpha})' = \alpha x^{\alpha - 1}$$

$$f'(x) = \underbrace{e^x}_{\substack{\vee \\ 0}} - \underbrace{(1-x)^{-2}}_{\substack{\vee \\ 0}}$$

f è derivabile $\forall x \neq 1$

$$f'(0) = 1 - (1)^{-2} = 0$$

voglio capire come è fatta f vicino a $x=0$.

$$f(x) = e^x - \overbrace{(1-x)^{-1}}$$

$x \to 0$

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$\alpha \in \mathbb{R}$

$$(1+x)^\alpha = 1 + \underline{\underline{\alpha}} \cdot \underline{\underline{x}} + \frac{1}{2}\alpha \cdot (\alpha-1)\, x^2 + o(x^2)$$

$$(1-x)^{-1} = [1 + (-x)]^{-1} = 1 + \overbrace{(-1)(-x)} +$$

$$+ \frac{1}{2}\underbrace{(-1)}\,(-1-1)\,(-x)^2 + o(-x)^2 = 1 + x +$$

$$+ \frac{1}{2} \cdot \underbrace{(-1) \cdot (-2)}\,(x^2) + o(x^2) = 1 + x + x^2 + o(x^2)$$

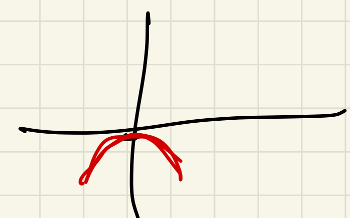
$$f(x) = e^x - (1-x)^{-1} = \qquad x \to 0$$

$$= 1 + x + \frac{1}{2}x^2 + o(x^2) - \left[1 + x + x^2 + o(x^2)\right]$$

$$= \cancel{1} + \cancel{x} + \frac{1}{2}x^2 + o(x^2) - \cancel{1} - \cancel{x} - x^2 + o(x^2)$$

$$= -\frac{1}{2}x^2 + o(x^2) \qquad \text{per } x \to 0$$

$$= -\frac{1}{2}x^2\left[1 + o(1)\right]$$



$\rightarrow$ $x=0$ pto di MAX LOCALE

$$\lim_{x\to+\infty} e^{x} - \frac{1}{1-x} = +\infty + 0 = +\infty$$

$e^x \to +\infty$

$\dfrac{1}{1-x} \to \dfrac{1}{-\infty} = 0$

a $+\infty$ NON HO AS. ORIZZONTALE

$$\lim_{x\to-\infty} e^{x} - \frac{1}{1-x} = 0+0 = 0$$

$e^{-\infty} = 0$

$\dfrac{1}{+\infty} = 0$

$y=0$

ASINT ORIZZ a $-\infty$

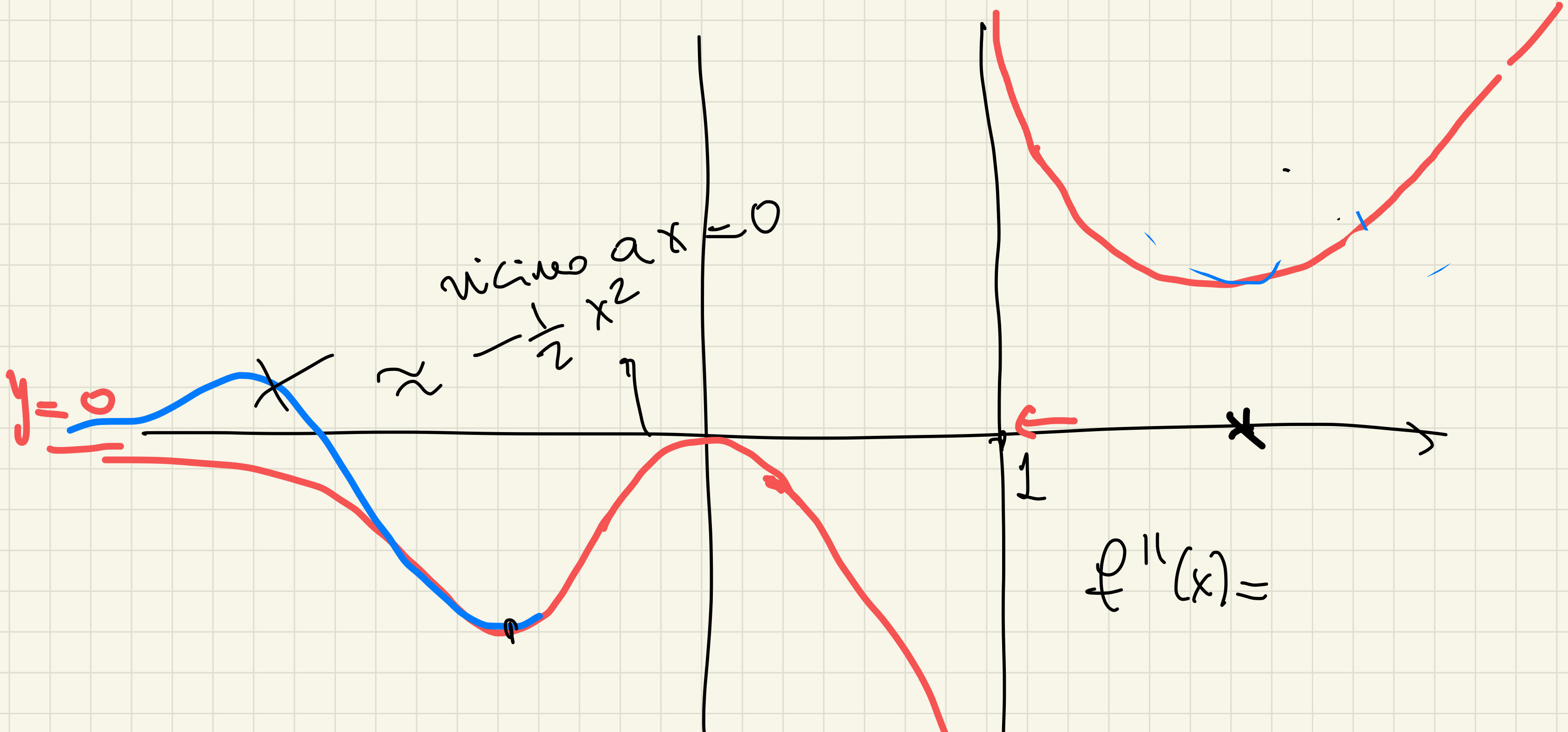
Cerco asintoto obliquo a $+\infty$

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{e^x - \frac{1}{(1-x)}}{x} =$$

$$= \lim_{x\to+\infty} \frac{e^x \left[1 - \frac{1}{(1-x)e^x}\right]}{x} = +\infty$$

$\to 1 - 0$

Non ho asintoto obliquo a $+\infty$



$f''(x) =$

per $x>1$ ho pto di MIN LOCALE

$x<0$ ho pto di MIN LOCALE

$x=1$

$$f'(x) = e^x - \frac{1}{(1-x)^2} = e^x - (1-x)^{-2}$$

$$f''(x) = e^x - \left[\underbrace{(-2)}\,(1-x)^{-2-1} \cdot \underbrace{(0-1)} \right] =$$

$$(x^\alpha)' = \alpha \cdot x^{\alpha-1}$$

$$= \underbrace{e^x}_{+} - \underbrace{\frac{2}{(1-x)^3}}_{+}$$

$x>1 \quad (1-x)<0$

$(1-x)^3<0$

$$-\frac{2}{(1-x)^3}>0$$

$$f''(x)>0$$

Es

Determinare al variare di $\alpha > 0$

$$\boxed{\lim_{x \to 0^+} \frac{x \cos x - \operatorname{sen} x}{e^{(x^\alpha)} - 1}}$$

$\boxed{N}$

$$\cos x = 1 - \frac{1}{2} x^2 + o(x^2)$$

$$\operatorname{sen} x = x - \frac{1}{6} x^3 + o(x^3)$$

$$x \cos x - \operatorname{sen} x = x\left[1 - \frac{1}{2} x^2 + o(x^2)\right] - \left(x - \frac{1}{6} x^3 + o(x^3)\right)$$

$$= \cancel{x} \underbrace{- \frac{1}{2} x^3} + \underbrace{x\, o(x^2)}_{o(x^3)} - \cancel{x} \underbrace{+ \frac{1}{6} x^3} + o(x^3) =$$

$$= \left(-\frac{1}{2}+\frac{1}{6}\right)x^3 + o(x^3) = -\frac{1}{3}x^3 + o(x^3)$$

$$= x^3\left[-\frac{1}{3} + o(1)\right]$$

D: $\underbrace{e^{x^\alpha} - 1}$ $\qquad \underline{\alpha > 0}$

$x \to 0^+$

$x^\alpha \to 0^+$

$$e^x = 1 + x + o(x)$$

$$e^{x^\alpha} = 1 + x^\alpha + o(x^\alpha)$$

$$e^{x^\alpha} - 1 = x^\alpha + o(x^\alpha) = x^\alpha\left[1 + o(1)\right]$$

$$\lim_{x\to 0^+} \frac{x^3\left[-\frac{1}{3}+o(1)\right]}{x^\alpha\left[1+o(1)\right]} =$$

$$= \begin{cases} \alpha = 3 & = -\frac{1}{3} \\ 3 > \alpha & = 0 \\ 3 < \alpha & = -\infty \end{cases}$$

Es Determinare al variare di $\alpha > 0$

$$\lim_{x \to +\infty} \underbrace{x^3}_{+\infty} \underbrace{\operatorname{arctg}\left(\frac{1}{x^2}\right)}_{\operatorname{arctg} 0 = 0} - \underbrace{x^\alpha}_{-(+\infty)}$$

$x \to 0 \qquad \operatorname{arctg} x = x - \frac{1}{3} x^3 + o(x^3)$

$\operatorname{arctg}\left(\frac{1}{x^2}\right)$

$x \to +\infty$

$$\operatorname{arctg}\left(\frac{1}{x^2}\right) = \frac{1}{x^2} - \frac{1}{3}\left(\frac{1}{x^2}\right)^3 + o\left(\left(\frac{1}{x^2}\right)^3\right)$$

$x \to +\infty$

$$= \frac{1}{x^2} - \frac{1}{3}\frac{1}{x^6} + o\left(\frac{1}{x^6}\right)$$

$$x^3 \cdot \operatorname{arctg} \frac{1}{x^2} =$$

$$= x^3 \cdot \Bigg[\underbrace{\frac{1}{x^2} - \frac{1}{3} \frac{1}{x^6} + o\left(\frac{1}{x^6}\right)}_{} \Bigg] =$$

in una somma con termini che vanno a 0 raccolgo quello di GRADO MINIMO

$$= x^{\cancel{3}} \cdot \frac{1}{\cancel{x^2}} \left[1 - \frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \right] =$$

$$= x \left[1 - \frac{1}{3} \frac{1}{x^4} + o\left(\frac{1}{x^4}\right) \right].$$

$$(+\infty)\cdot(0)$$

$$\lim_{x\to+\infty} \underbrace{x^3 \operatorname{arctg}\frac{1}{x^2}} - x^{\alpha} =$$

$$= \lim_{x\to+\infty} \underbrace{x^{1}}_{+\infty} \underbrace{\left[1-\frac{1}{3}\,\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right]}_{1+0+0=1} - x^{\alpha}$$

(quando ho somme di termini che tendono a ∞ raccolgo quello di grado MAGGIORE

$\boxed{\text{Se } \alpha > 1}$

$$\lim_{x\to+\infty} x\left[1-\frac{1}{3}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right]-x^\alpha =$$

$$= \lim_{x\to+\infty} \underbrace{x^\alpha}_{\to +\infty} \left[\underbrace{\frac{x}{x^\alpha}}_{\to 0}\underbrace{\left(1-\frac{1}{3}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right)}_{\to 1} - 1\right]$$

$$= +\infty \cdot (-1) = -\infty .$$

se $\alpha = 1$

$$\lim_{x\to+\infty} x\left[1-\frac{1}{3}\cdot\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right]-x=$$

$$=\lim_{x\to+\infty} x\left[\cancel{1}-\frac{1}{3}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\cancel{-1}\right]=$$

$$=\lim_{x\to+\infty} x\left[-\frac{1}{3}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right]=\lim_{x\to+\infty}\cancel{x}\cdot\frac{1}{x^{\cancel{4}\,3}}\left[-\frac{1}{3}+o(1)\right]$$

$$=0\cdot\left(-\frac{1}{3}\right)=0$$

Se $\alpha < 1$

$$\lim_{x\to+\infty} x\cdot\left[1-\frac{1}{3}\frac{1}{x^4}+o\left(\frac{1}{x^4}\right)\right]-x^{\alpha}$$

$$=\lim_{x\to+\infty} \underbrace{x}_{+\infty}\left[1-\underbrace{\frac{1}{3}\frac{1}{x^4}}_{0}+\underbrace{o\left(\frac{1}{x^4}\right)}_{0}-\underbrace{\frac{x^{\alpha}}{x}}_{0}\right]$$

$$=+\infty\cdot 1=+\infty.$$

Es Calcolare al variare di $\alpha > 0$

$$\lim_{n \to +\infty} \frac{n^{\alpha} \cdot \left[\lg\left(1 + \frac{1}{2n^2}\right) - 1 + \cos\left(\frac{1}{n}\right) \right]}{\left[\underbrace{\left(\operatorname{sen}\left(\frac{1}{n}\right)\right) - \frac{1}{n}}_{0 - 0} \right]}$$

$$\operatorname{sen}\left(\frac{1}{n}\right) - \frac{1}{n}$$

$n \to +\infty$

$\frac{1}{n} \to 0$

$\left(\frac{1}{n}\right)^3 = \frac{1}{n^3}$

$x \to 0$

$$\operatorname{sen} x = x - \frac{1}{6} x^3 + o(x^3) \qquad x \to 0$$

$$\operatorname{sen}\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{6}\left(\frac{1}{n}\right)^3 + o\left(\frac{1}{n}\right)^3 \qquad n \to +\infty$$

$$\left[\operatorname{sen}\left(\frac{1}{n}\right)\right] - \frac{1}{n} = \cancel{\frac{1}{n}} - \frac{1}{6}\left(\frac{1}{n}\right)^3 + o\left(\frac{1}{n}\right)^3 - \cancel{\frac{1}{n}}$$

$$= \frac{1}{n^3}\left[-\frac{1}{6} + o(1)\right] = n^{-3}\left[-\frac{1}{6} + o(1)\right]$$

$$\underbrace{\lg\left(1+\frac{1}{2n^2}\right) - 1 + \cos\left(\frac{1}{n}\right)}$$

$n \to +\infty$

$x \to 0$

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$$

$$\cos\left(\frac{1}{n}\right) = 1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + o\left(\frac{1}{n}\right)^2$$

$x \to 0$

$$\lg(1+x) = x + o(x)$$

$$\lg\left(1+\frac{1}{2n^2}\right) = \frac{1}{2n^2} + o\left(\frac{1}{\cancel{2}n^2}\right)$$

$$\cancel{\frac{1}{2n^2}} + o\left(\frac{1}{n^2}\right) \cancel{-1} + \cancel{1} \cancel{- \frac{1}{2}\left(\frac{1}{n^2}\right)} + o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n^2}\right)$$

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2) \qquad \cos\frac{1}{n} = 1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + o\left(\frac{1}{n}\right)^2$$

$$\log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$

$$\log\left(1+\frac{1}{2n^2}\right) = \frac{1}{2n^2} - \frac{1}{2}\left(\frac{1}{2n^2}\right)^2 + o\left(\left(\frac{1}{\cancel{2}n^2}\right)^2\right) =$$

$$= \boxed{\frac{1}{2n^2} - \frac{1}{2}\cdot\left(\frac{1}{4n^4}\right) + o\left(\frac{1}{n^4}\right)}$$

$$\log\left(1+\frac{1}{2n^2}\right) - 1 + \cos\frac{1}{n} = \cancel{\frac{1}{2n^2}} - \frac{1}{8}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) \cancel{-1} + \cancel{1}$$

$$\cancel{-\frac{1}{2}\frac{1}{n^2}} + o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n^2}\right)$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$\cos\left(\frac{1}{n}\right) = 1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + \frac{1}{24}\left(\frac{1}{n}\right)^4 + o\left(\frac{1}{n^4}\right)$$

$$\lg\left(1+\frac{1}{2n^2}\right) - 1 + \cos\left(\frac{1}{n}\right) =$$

$$= \cancel{\frac{1}{2n^2}} - \frac{1}{8}\,\frac{1}{n^4} + o\left(\frac{1}{n}\right)^4 \cancel{-1} + \cancel{1} \cancel{- \frac{1}{2}\,\frac{1}{n^2}} +$$

$$+ \frac{1}{24}\,\frac{1}{n^4} + o\left(\frac{1}{n}\right)^4 = \left(-\frac{1}{8} + \frac{1}{24}\right)\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

$$\lg\left(1+\frac{1}{2n^2}\right)-1+\cos\frac{1}{n}=$$

$$=-\frac{1}{12}\,\frac{1}{n^4}+o\left(\frac{1}{n^4}\right)=\frac{1}{n^4}\left[-\frac{1}{12}+o(1)\right]$$

$$=n^{-4}\left[-\frac{1}{12}+o(1)\right]$$

$$\left(\frac{1}{n}\right)^4=\frac{1}{n^4}=n^{-4}$$

$$\lim_{n\to+\infty} \frac{n^{\alpha}\left[\log\left(1+\frac{1}{2n^2}\right)-1+\cos\frac{1}{n}\right]}{\operatorname{sen}\frac{1}{n}-\frac{1}{n}} =$$

$$= \lim_{n\to+\infty} \frac{n^{\alpha}\, n^{-4}\left(-\frac{1}{12}+o(1)\right)}{n^{-3}\left(-\frac{1}{6}+o(1)\right)} =$$

$$= \lim_{n\to+\infty} n^{\overbrace{\alpha-4+3}^{\alpha-1}} \frac{\left(-\frac{1}{12}+o(1)\right)}{\left(-\frac{1}{6}+o(1)\right)} = \begin{cases} \alpha=1 & = \dfrac{-\frac{1}{12}}{-\frac{1}{6}} = \dfrac{1}{2} \\ \alpha>1 & =+\infty \\ \alpha<1 & =0 \end{cases}$$

(abbiamo visto che

$$n^{\alpha} \cdot \frac{\left(\log\left(1+\frac{1}{2n^2}\right) - 1 + \cos\frac{1}{n}\right)}{\operatorname{sen}\frac{1}{n} - \frac{1}{n}}$$

per $n \to +\infty$

$$= n^{\alpha-1} \underbrace{\frac{\left[-\frac{1}{12} + o(1)\right]}{\left(-\frac{1}{6} + o(1)\right)}}_{\frac{1}{2}} \approx n^{\alpha-1} = \frac{1}{n^{1-\alpha}}$$

Es) Det al variare di $\alpha > 0$

$$\lim_{n \to +\infty} \underbrace{\left(n^{\alpha} - \operatorname{arctg} n + \lg n \right)}_{} \cdot \underbrace{\left[\sqrt[3]{1+\frac{1}{n}} - e^{\frac{1}{3n}} \right]}_{}$$

$$\underset{\downarrow\ +\infty}{n^{\alpha}} - \underset{\downarrow\ \frac{\pi}{2}}{\underbrace{\operatorname{arctg} n}} + \underset{\downarrow\ +\infty}{\underbrace{\lg n}} = n^{\alpha} \left[1 - \underset{\downarrow\ 0}{\frac{\operatorname{arctg} n}{n^{\alpha}}} + \underset{\downarrow\ 0}{\frac{\lg n}{n^{\alpha}}} \right]$$

$$= n^{\alpha} \left(1 + o(1) \right)$$

$$\sqrt[3]{1+\frac{1}{n}} - e^{\frac{1}{3n}}$$

$n \to +\infty \qquad \frac{1}{n} \to 0$

$\frac{1}{3n} \to 0$

$x \to 0$

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x + \frac{1}{2} \cdot \frac{1}{3}\left(\frac{1}{3} - 1\right)x^2 + o(x^2)$$

$x \to 0$

$$= 1 + \frac{1}{3}x + \frac{1}{\cancel{2}} \cdot \frac{1}{3} \cdot \left(-\frac{\cancel{2}}{3}\right)x^2 + o(x^2) =$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + o(x^2)$$

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{1}{2}\alpha(\alpha-1)x^2 + o(x^2)$$

$$\sqrt[3]{1+\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{\frac{1}{3}} = 1 + \frac{1}{3} \cdot \frac{1}{n} - \frac{1}{9}\left(\frac{1}{n}\right)^2 + o\left(\frac{1}{n}\right)^2$$

$$e^{x} = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^{\frac{1}{3n}} = 1 + \frac{1}{3n} + \frac{1}{2}\cdot\left(\frac{1}{3n}\right)^2 + o\left(\frac{1}{\cancel{3}n}\right)^2 =$$

$$= 1 + \frac{1}{3}\cdot\frac{1}{n} + \frac{1}{2}\cdot\frac{1}{9}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) =$$

$$= 1 + \frac{1}{3}\frac{1}{n} + \frac{1}{18}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

$$\sqrt[3]{1+\frac{1}{n}} - e^{\frac{1}{3n}} =$$

$$= 1 + \frac{1}{3}\frac{1}{n} - \frac{1}{9}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) - \left[1 + \frac{1}{3}\frac{1}{n} + \frac{1}{18}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right]$$

$$= \cancel{1} + \cancel{\frac{1}{3}\frac{1}{n}} \underbrace{- \frac{1}{9}\frac{1}{n^2}} + o\left(\frac{1}{n^2}\right) - \cancel{1} - \cancel{\frac{1}{3}\frac{1}{n}} \underbrace{- \frac{1}{18}\frac{1}{n^2}} + o\left(\frac{1}{n^2}\right)$$

$$= \left(-\frac{1}{9} - \frac{1}{18}\right)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) = -\frac{1}{6}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right) =$$

$$= \frac{1}{n^2}\left(-\frac{1}{6} + o(1)\right)$$

$$\lim_{n\to+\infty} \overbrace{n^{\alpha}\cdot\left(1+o(1)\right)}\cdot\underbrace{\frac{1}{n^2}\left(-\frac{1}{6}+o(1)\right)} =$$

$$= \lim_{n\to+\infty} \frac{n^{\alpha}}{n^2}\underbrace{\left(1+o(1)\right)}\underbrace{\left(-\frac{1}{6}+o(1)\right)} =$$

$$= \begin{cases} \alpha = 2 & = -\dfrac{1}{6} \\ \alpha > 2 & = (+\infty)\cdot 1\cdot\left(-\dfrac{1}{6}\right) = -\infty \\ \alpha < 2 & = 0 \end{cases}$$