Domani 10.30 → 13.30 (con pause)

$o(x^N) = \{ f$ tali che $\lim_{x\to 0} \frac{f(x)}{x^N} = 0 \}$

$o(1) = \{ f$ tali che $\lim_{x\to 0} f(x) = 0 \}$

Teorema di Taylor

Se f è derivabile infinite volte in un intervallo che contiene $x_0 = 0$, allora

$\forall N \in \mathbb{N}$ $\exists P_N(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots + \frac{1}{N!} f^{(N)}(0) x^N$

tale che $f(x) = P_N(x) + o(x^N)$

$$\Updownarrow \quad \lim_{x\to 0} \frac{f(x) - P_N(x)}{x^N} = 0$$

"Regole di calcolo con $o(x^n)$"

(1) $o(x^n) \subseteq o(x^m)$ $n > m$

$\boxed{n > m}$

$$\underbrace{o(x^n)} + \underbrace{o(x^m)} \subseteq o(x^m) + o(x^m) = o(x^m)$$

$$\boxed{o(x^n) + o(x^m) = o(x^m)}$$ se $n > m$

$$o(x^2) + o(x^3) = o(x^2)$$

$\boxed{\text{perché } o(x^3) \subseteq o(x^2)}$

2) $$x^k \underbrace{\sigma(x^m)} = \sigma(x^{k+m})$$

$$\sigma(x^k)\,\sigma(x^m) = \sigma(x^{k+m})$$

$$\left[\sigma(x^m)\right]^k = \sigma(x^{mk})$$

Es

$$\lim_{x\to 0} \frac{x - \sin x + x^2}{1 - \cos x + x^3}$$

Sostituisco sen x e cos x con i loro polinomi. Ma di che grado?

NUMERATORE $\underbrace{x - \overbrace{\sin x} + x^2}$

$x + x^2$ è pol. di grado 2 → prendo POLINOMIO di grado 2 di sin x

$$\sin x = P_2(x) + o(x^2)$$

$$P_2(x) = 0 + x + 0$$

$$\boxed{\sin x = x + o(x^2)}$$

$\sin x = x + o(x^2)$ $\qquad P_2(x) = x = P_1(x)$

$N: \quad x - \sin x + x^2 =$

$= x - (x + \underline{o(x^2)}) + x^2 =$

$= \cancel{x} - \cancel{x} + \underbrace{o(x^2)} + \underbrace{x^2}$ $\qquad x^2 \cdot o(1) = o(x^2)$

$o(x^2)$ è un insieme di funzioni!

$-o(x^2) = o(x^2) = a\,o(x^2) \quad \forall a \in \mathbb{R}.$

$= x^2 \left[\underbrace{o(1) + 1}_{\downarrow \atop 0 + 1} \right]$ $\qquad o(1) \xrightarrow{x \to 0} 0$

$$D: \quad 1 - \cos x + x^3$$

$$P_3(x) = 1 + 0 - \frac{1}{2}x^2 + 0$$

$$x^2 \cdot o(x) = o(x^3)$$

$$\boxed{\cos x = 1 - \frac{1}{2}x^2 + o(x^3)}$$

$$1 - \cos x + x^3 = 1 - \left(1 - \frac{1}{2}x^2 + o(x^3)\right) + x^3 =$$

$$= \cancel{1} - \cancel{1} + \frac{1}{2}x^2 + o(x^3) + x^3 =$$

$$= x^2 \left[\frac{1}{2} + \underbrace{o(x)}_{\to 0} + \underbrace{x}_{\to 0} \right]$$

$$\lim_{x \to 0} \frac{\cancel{x^2}\,[o(1)+1]}{\cancel{x^2}\,[\frac{1}{2}+o(x)+x]} = \frac{0+1}{\frac{1}{2}+0+0} =$$

$$= \frac{1}{\frac{1}{2}} = 2$$

Es

$$\lim_{x\to 0} \frac{e^{x^2}-1-x^2}{\lg(1-x)+x}$$

N $\quad e^{x^2}-1-x^2$ $\qquad e^{(x^2)}$

$$\boxed{e^{x} = 1 + x + \frac{1}{2}x^2 + o(x^2)}$$

$x \to 0$

$\downarrow$

$x^2 \to 0$

$$e^{x^2} = 1 + x^2 + \frac{1}{2}(x^2)^2 + o((x^2)^2)$$

$$= 1 + x^2 + \frac{1}{2}x^4 + o(x^4)$$

N: $e^{x^2} - 1 - x^2$

e^{x^2} lo voglio sostituire con polinomio di grado almeno 2 nella variabile x

$$\boxed{e^{x} = 1 + x + o(x)}$$

$$\boxed{e^{x^2} = 1 + x^2 + o(x^2)}$$

N: $e^{x^2} - 1 - x^2 = \cancel{1} \cancel{+ x^2} + o(x^2) \cancel{- 1} - \cancel{x^2} = o(x^2)$

$\Rightarrow o(x^2)$ è troppo generico, torno indietro e vado al grado successivo del polinomio

$$e^{x} = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^{x^2} = 1 + x^2 + \frac{1}{2}(x^2)^2 + o((x^2)^2)$$

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + o(x^4)$$

$N: e^{x^2} - 1 - x^2 = \cancel{1} + \cancel{x^2} + \frac{1}{2}x^4 + o(x^4) \cancel{-1} \cancel{-x^2}$

$$= \frac{1}{2}x^4 + o(x^4) = x^4\left(\frac{1}{2} + o(1)\right)$$

D: $\underbrace{\lg(1-x)} + \underbrace{x}$

$x \to 0$

$$\boxed{\lg(1+x) = x + o(x)}$$

se $x \to 0$ anche $-x \to 0$

$$\lg(1-x) = \lg(1+(-x)) = -x + o(-x) =$$

$$= -x + o(x)$$

$$\lg(1-x) + x = \cancel{-x} + o(x) \cancel{+x} = o(x).$$ TROPPO GENERICO

$$\lg(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$

$$\lg(1+(-x)) = (-x) - \frac{1}{2}(-x)^2 + o((-x)^2) = \triangleright$$

$$\underline{\lg(1+(-x))} = (-x) - \frac{1}{2}\underbrace{(-x)}^{2} + o((-x)^2)$$

$$= \underbrace{-x - \frac{1}{2}x^2 + o(x^2)}$$

$$\lg(1-x) \underline{+x} = \cancel{-x} - \underline{\frac{1}{2}x^2} + o(\underline{x^2}) \; \underline{+\cancel{x}} =$$

$$= x^2\left(-\frac{1}{2} + o(1)\right)$$

$$\lim_{x\to 0} \frac{x^{\cancel{4}\,2}\left(\frac{1}{2}+o(1)\right)}{\cancel{x^2}\left(-\frac{1}{2}+o(1)\right)} = \frac{0 \cdot \frac{1}{2}}{-\frac{1}{2}} = 0$$

$\to \frac{1}{2} + 0 = \frac{1}{2}$

$\to -\frac{1}{2} + 0 = -\frac{1}{2}$

0

Es Determinare al variare di $\alpha > 0$

$$\lim_{x \to 0^+} \frac{\sin(x) - x^\alpha}{\sqrt{1+x^2} - 1}$$

NUMERATORE

$$\operatorname{sen} x = x - \frac{1}{6}x^3 + o(x^3)$$

$$\sin x - x^\alpha = x - \frac{1}{6}x^3 + o(x^3) - x^\alpha$$

$\alpha = 1$ $\quad x^3\left(-\frac{1}{6} + o(1)\right)$

$\alpha > 1$ $\quad x\left[1 - \frac{1}{6}x^2 + o(x^2) - x^{\alpha-1}\right]$ $\quad \downarrow 0$

$0 < \alpha < 1$ $\quad x^\alpha\left[x^{1-\alpha} - \frac{1}{6}x^{3-\alpha} + o(x^{3-\alpha}) - 1\right]$

$(1+x^2)^{\frac{1}{2}} - 1$

$(1+x)^k$

è ben definita $\forall k$

& $1+x>0$

$x>-1$

$x \to 0$

$$(1+x)^k = \underset{f(0)}{1} + \underbrace{\underset{f'(0)}{k} \cdot x} + \frac{1}{2} \underset{f''(0)}{k(k-1)} x^2 + o(x^2)$$

$f(x) = (1+x)^k$ $\qquad f(0) = 1$ $\qquad \forall k \in \mathbb{R}$

$f'(x) = k(1+x)^{k-1}$

$f''(x) = k \cdot (k-1)(1+x)^{k-2}$

$$\left((1+x)^{\frac{1}{2}}\right) = 1 + \overset{k}{\frac{1}{2}} x + \frac{1}{2} \overset{k}{\frac{1}{2}} \overset{k-1}{\left(\frac{1}{2}-1\right)} x^2 + o(x^2) \qquad k = \frac{1}{2}$$

$$\sqrt{1+x}$$

$$(1+x)^k = 1 + kx + \frac{1}{2} k(k-1) x^2 + o(x^2)$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2} x + o(x)$$

$$(1+x^2)^{\frac{1}{2}} = 1 + \frac{1}{2} x^2 + o(x^2)$$

$$D: (1+x^2)^{\frac{1}{2}} - 1 = \cancel{1} + \frac{1}{2} x^2 + o(x^2) \cancel{-1} = x^2\left(\frac{1}{2} + o(1)\right)$$

Calcolo del limite

$\alpha = 1$

$$\lim_{x\to 0^+} \frac{x^3\left(-\frac{1}{6}+o(1)\right)}{\cancel{x^2}\left(\frac{1}{2}+o(1)\right)} = 0$$

$x^3 \to 0$, $\left(-\frac{1}{6}+o(1)\right) \to -1/6$, $\left(\frac{1}{2}+o(1)\right) \to 1/2$

$\alpha > 1$

$$\lim_{x\to 0^+} \frac{\cancel{x}\left[1-\frac{1}{6}x^2+o(x^2)-x^{\alpha-1}\right]}{\cancel{x^2}\left(\frac{1}{2}+o(1)\right)} = \frac{+\infty\cdot 1}{\frac{1}{2}} = +\infty$$

$\left[1-\frac{1}{6}x^2+o(x^2)-x^{\alpha-1}\right] \to 1$, $\left(\frac{1}{2}+o(1)\right) \to \frac{1}{2}$

$\alpha < 1$

$$\lim_{x\to 0^+} \frac{x^\alpha}{x^2}\cdot\frac{\left[x^{1-\alpha}-\frac{1}{6}x^{3-\alpha}+o(x^{3-\alpha})-1\right]}{\left(\frac{1}{2}+o(1)\right)} = \frac{+\infty(-1)}{1/2} = -\infty$$

$\left[x^{1-\alpha}-\frac{1}{6}x^{3-\alpha}+o(x^{3-\alpha})-1\right] \to -1$, $\left(\frac{1}{2}+o(1)\right) \to 1/2$

Es

$$\lim_{\substack{x \to 0^+ \\ x>0}} \frac{\cos(\sqrt{x}) - e^x + \frac{1}{2}x^2}{\operatorname{tg} x - \sin x}$$

$\boxed{\text{NUMERATORE}}$

$\underbrace{x \to 0^+}$ $\to \sqrt{x} \to 0$

$\cos x$ = polinomio di grado 4 $= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$

$\cos\sqrt{x} = 1 - \frac{1}{2}(\sqrt{x})^2 + \frac{1}{24}(\sqrt{x})^4 + o(\sqrt{x})^4$

$\stackrel{!}{=} 1 - \frac{1}{2}x + \frac{1}{24}x^2 + o(x^2)$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

N1

$$\underbrace{\cos\sqrt{x}} - \underbrace{e^x} + \frac{1}{2}x^2 =$$

$$= \underbrace{1 - \frac{1}{2}x + \frac{1}{24}x^2 + o(x^2)} - \underbrace{\left(1 + x + \frac{x^2}{2} + o(x^2)\right)}$$

$$+ \frac{1}{2}x^2 = \cancel{1} \underbrace{-\frac{1}{2}x} + \frac{1}{24}x^2 + o(x^2) - \cancel{1} \underbrace{-x} - \cancel{\frac{1}{2}x^2} +$$

$$+ o(x^2) + \cancel{\frac{1}{2}x^2} =$$

$$= -\frac{3}{2}x + \frac{1}{24}x^2 + o(x^2) = x\left[-\frac{3}{2} + \frac{1}{24}x + o(x)\right]$$

D: $\operatorname{tg} x - \sin x$

pol. di grado 3

$$\operatorname{tg} x = \cancel{0} + x + \frac{1}{3!}\, 2x^3 + o(x^3)$$

$$= x + \frac{1}{6}\, 2x^3 + o(x^3)$$

$$= \underbrace{x + \frac{1}{3}x^3 + o(x^3)}$$

$$f(x) = \operatorname{tg} x$$

$$\boxed{f'(x) = 1 + (\operatorname{tg} x)^2}$$

$$f''(x) = 2\operatorname{tg} x\,(1 + (\operatorname{tg} x)^2)$$

$$f^{(3)}(x) = 2(1+(\operatorname{tg} x)^2)(1+(\operatorname{tg} x)^2) + 2\operatorname{tg} x \cdot 2\operatorname{tg} x(1+(\operatorname{tg} x)^2)$$

$$\sin x = x - \frac{1}{6}x^3 + o(x^3)$$

$$\operatorname{tg} x - \sin x = x + \frac{1}{3}x^3 + o(x^3) - \left(x - \frac{1}{6}x^3 + o(x^3)\right) =$$

$$= \cancel{x} + \frac{1}{3}x^3 + o(x^3) \cancel{-x} + \frac{1}{6}x^3 + o(x^3) = \frac{1}{2}x^3 + o(x^3) = x^3\left(\frac{1}{2} + o(1)\right)$$

$$\lim_{x \to 0^+} \frac{\cancel{x}\left(-\frac{3}{2} + \frac{1}{24}x + o(x)\right)}{x^{\cancel{3}\,2}\left(\frac{1}{2} + o(1)\right)} = \frac{+\infty \cdot \left(-\frac{3}{2}\right)}{\frac{1}{2}} = -\infty$$

$\to -\frac{3}{2} + 0 + 0$

$\frac{1}{x^2} \to +\infty$