

Domani 10.30 → 13.30 (con pausa)

$$o(x^N) = \left\{ f \text{ tali che } \lim_{x \rightarrow 0} \frac{f(x)}{x^N} = 0 \right\}$$

$$o(1) = \left\{ f \text{ tali che } \lim_{x \rightarrow 0} f(x) = 0 \right\}$$

## Teorema di Taylor

Se  $f$  è derivabile infinite volte in un intervallo che contiene  $x_0 = 0$ , allora

$$\forall N \in \mathbb{N} \quad \exists P_N(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots + \frac{1}{N!} f^{(N)}(0)x^N$$

tali che  $f(x) = P_N(x) + o(x^N)$

$$\Leftrightarrow \lim_{x \rightarrow 0} \frac{f(x) - P_N(x)}{x^N} = 0$$

# "Regole di calcolo con $\mathcal{O}(x^n)$ "

$$(1) \quad \mathcal{O}(x^n) \subseteq \mathcal{O}(x^m) \quad n > m$$

$$n > m$$

$$\mathcal{O}(x^n) + \mathcal{O}(x^m) \subseteq \mathcal{O}(x^n) + \mathcal{O}(x^m) = \mathcal{O}(x^m)$$

$$\mathcal{O}(x^n) + \mathcal{O}(x^m) = \mathcal{O}(x^m) \quad \text{se } n > m$$

$$\mathcal{O}(x^2) + \mathcal{O}(x^3) = \mathcal{O}(x^2)$$

$$\text{perché } \mathcal{O}(x^3) \subseteq \mathcal{O}(x^2)$$

$$2) \quad x^k \underbrace{\sigma(x^m)} = \sigma(x^{k+m})$$

$$\sigma(x^k) \sigma(x^m) = \sigma(x^{k+m})$$

$$[\sigma(x^m)]^k = \sigma(x^{mk})$$

Es

$$\lim_{x \rightarrow 0} \frac{x - \sin x + x^2}{1 - \cos x + x^3}$$

Lo sviluppo  
polinomico  
di  $\sin x$  e  $\cos x$  con i loro  
Ma di che grado?

NUMERATORE

$$x - \sin x + x^2$$

$x + x^2$  è pol. di grado 2  $\rightarrow$  prendo polinomio  
di grado 2 di  $\sin x$

$$P_2(x) = 0 + x + 0$$

$$\sin x = P_2(x) + o(x^2)$$
$$\boxed{\sin x = x + o(x^2)}$$

$$\lim x = x + o(x^2)$$

$$P_2(x) = x = P_1(x)$$

$$N: x - \lim x + x^2 =$$

$$= x - (x + o(x^2)) + x^2 =$$

$$= \cancel{x} - \cancel{x} + \underbrace{o(x^2)} + \underbrace{x^2}$$

$$x^2 \cdot o(1) = o(x^2)$$

$o(x^2)$  è un insieme di funzioni!

$$- o(x^2) = o(x^2) = a o(x^2) \quad \forall a \in \mathbb{R}$$

$$= x^2 [o(1) + 1]$$

$$o(1) \xrightarrow{x \rightarrow 0} 0$$

$$0 + 1$$

$$D: \underline{1} - \cos x + \underline{x^3}$$

$$P_3(x) = 1 + 0 - \frac{1}{2}x^2 + 0$$

$$x^2 \cdot o(x) = o(x^3)$$

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^3)$$

$$1 - \cos x + x^3 = 1 - \left(1 - \frac{1}{2}x^2 + o(x^3)\right) + x^3 =$$

$$= \cancel{1} - \cancel{1} + \frac{1}{2}x^2 + \underline{o(x^3)} + x^3 =$$

$$= x^2 \left[ \frac{1}{2} + \underbrace{o(x)}_{\downarrow 0} + \underbrace{x}_{\downarrow 0} \right]$$

$$\lim_{x \rightarrow 0} \frac{\cancel{x^2} [\mathcal{O}(1) + 1]}{\cancel{x^2} \left[ \frac{1}{2} + \mathcal{O}(x) + x \right]} = \frac{0 + 1}{\frac{1}{2} + 0 + 0} =$$

$$= \frac{1}{\frac{1}{2}} = 2$$

es

lim  
 $x \rightarrow 0$

$$\frac{e^{x^2} - 1 - x^2}{\lg(1-x) + x}$$

N

$$\frac{e^{x^2} - 1 - x^2}{e^{x^2}}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$x \rightarrow 0$   
 $\downarrow$   
 $x^2 \rightarrow 0$

$$e^{x^2} = 1 + x^2 + \frac{1}{2}(x^2)^2 + o((x^2)^2)$$
$$= 1 + x^2 + \frac{1}{2}x^4 + o(x^4)$$



$$N: e^{x^2} - 1 - x^2$$

$e^{x^2}$  lo voglio sostituire con polinomio di grado almeno 2 nella variabile  $x$

$$e^x = 1 + x + o(x)$$

$$e^{x^2} = 1 + \underline{x^2} + o(x^2)$$

$$N: e^{x^2} - 1 - x^2 = \cancel{1 + x^2} + o(x^2) - \cancel{1 - x^2} = o(x^2)$$

$\Rightarrow o(x^2)$  è troppo generico, forse indietro

e vede al grado necessario del polinomio

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^{x^2} = 1 + x^2 + \frac{1}{2}(x^2)^2 + o((x^2)^2)$$

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + o(x^4)$$

$$\begin{aligned} R : e^{x^2} - 1 - x^2 &= \cancel{1 + x^2} + \frac{1}{2}x^4 + o(x^4) - \cancel{1 - x^2} \\ &= \frac{1}{2}x^4 + o(x^4) = x^4 \left( \frac{1}{2} + o(1) \right) \end{aligned}$$

$$D: \underbrace{\lg(1-x)} + x$$

$x \rightarrow 0$

$$\lg(1+x) = x + o(x)$$

se  $x \rightarrow 0$  anche  
 $-x \rightarrow 0$

$$\begin{aligned} \lg(1-x) &= \lg(1+(-x)) = -x + o(-x) = \\ &= -x + o(x) \end{aligned}$$

$$\lg(1-x) + x = \cancel{-x} + o(x) + \cancel{x} = o(x). \quad \text{TROPPO GENERICO}$$

$$\lg(1+x) = x - \frac{1}{2}x^2 + o(x^2)$$

$$\lg(1+(-x)) = (-x) - \frac{1}{2}(-x)^2 + o((-x)^2) = \Delta$$

$$\begin{aligned} \lg(1 + (-x)) &= (-x) = \frac{1}{2} (-x)^2 + o((-x)^2) \\ &= -x - \frac{1}{2} x^2 + o(x^2) \end{aligned}$$

$$\lg(1-x) + x = \cancel{-x} - \frac{1}{2} x^2 + o(x^2) + \cancel{x} =$$

$$= x^2 \left( -\frac{1}{2} + o(1) \right)$$

$$\lim_{x \rightarrow 0} \frac{x^4 \left( \frac{1}{2} + o(1) \right)}{x^2 \left( -\frac{1}{2} + o(1) \right)} = \frac{0 \cdot \frac{1}{2}}{-\frac{1}{2}} = 0$$

(Note: In the original image, a red arrow points to the  $x^4$  term, and blue arrows indicate the limits of the terms in parentheses:  $\frac{1}{2} + o(1) \rightarrow \frac{1}{2} + 0 = \frac{1}{2}$  and  $-\frac{1}{2} + o(1) \rightarrow -\frac{1}{2} + 0 = -\frac{1}{2}$ )

ES Determinare al valore di  $\alpha > 0$

$$\lim_{x \rightarrow 0^+} \frac{\sin(x) - x^\alpha}{\sqrt{1+x^2} - 1}$$

NUMERATORE

$$\text{sen } x = x - \frac{1}{6}x^3 + o(x^3)$$

$$\sin x - x^\alpha = x - \frac{1}{6}x^3 + o(x^3) - x^\alpha$$

$0 < \alpha < 1$

$\alpha = 1$   $x^3 \left( -\frac{1}{6} + o(1) \right)$

$\alpha > 1$   $x \left[ 1 - \frac{1}{6}x^2 + o(x^2) - x^{\alpha-1} \right]$

$x^\alpha \left[ x^{1-\alpha} - \frac{1}{6}x^{3-\alpha} + o(x^{3-\alpha}) - 1 \right]$

$$(1+x^2)^{\frac{1}{2}} - 1$$

$$(1+x)^k$$

$\bar{e}$  bzw. definiert  
 $\forall k$   
&  $1+x > 0$   
 $x > -1$

$$x \rightarrow 0$$

$$(1+x)^k = \underbrace{1}_{f(0)} + \underbrace{k \cdot x}_{f'(0)} + \underbrace{\frac{1}{2} k(k-1) x^2}_{f''(0)} + o(x^2)$$

$$f(x) = (1+x)^k \quad f(0) = 1 \quad \forall k \in \mathbb{R}$$

$$f'(x) = k(1+x)^{k-1}$$

$$f''(x) = k \cdot (k-1) (1+x)^{k-2}$$

$$\underbrace{\left( (1+x)^{\frac{1}{2}} \right)}_{\sqrt{1+x}} = 1 + \frac{1}{2}x + \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} - 1 \right) x^2 + o(x^2) \quad k = \frac{1}{2}$$

$$(1+x)^k = 1 + kx + \frac{1}{2} k(k-1)x^2 + o(x^2)$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + o(x)$$

$$(1+x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + o(x^2)$$

$$D: (1+x^2)^{\frac{1}{2}} - 1 = \cancel{1} + \frac{1}{2}x^2 + o(x^2) - \cancel{1} = x^2 \left( \frac{1}{2} + o(1) \right)$$

# calcolo del limite

$\alpha = 1$

$$\lim_{x \rightarrow 0^+} \frac{x^3 \left( -\frac{1}{6} + o(1) \right)}{x^2 \left( \frac{1}{2} + o(1) \right)} = 0$$

*Annotations: Red arrows point from  $0$  to  $x^3$  and from  $-\frac{1}{6}$  to the term in parentheses. A red arrow points from  $\frac{1}{2}$  to the term in parentheses. A red arrow points from  $1$  to the denominator.*

$\alpha > 1$

$$\lim_{x \rightarrow 0^+} \frac{x \left[ 1 - \frac{1}{6} x^2 + o(x^2) - x^{\alpha-1} \right]}{x^2 \left( \frac{1}{2} + o(1) \right)} = \frac{+\infty \cdot 1}{\frac{1}{2}} = +\infty$$

*Annotations: Blue circles around  $x$  and  $x^2$ . Orange circles around the numerator and denominator. Orange arrows point from  $1$  to the term in parentheses and from  $\frac{1}{2}$  to the term in parentheses.*

$\alpha < 1$

$$\lim_{x \rightarrow 0^+} \frac{x^\alpha}{x^2 \left( \frac{1}{2} + o(1) \right)} = \frac{+\infty (-1)}{\frac{1}{2}} = -\infty$$

*Annotations: Yellow circles around  $x^\alpha$  and  $+\infty$ . Red circles around the numerator and denominator. Red arrows point from  $-1$  to the term in parentheses and from  $\frac{1}{2}$  to the term in parentheses.*



Es

lim

$x \rightarrow 0^+$

$x > 0$

$$\frac{\cos(\sqrt{x}) - e^x + \frac{1}{2}x^2}{\operatorname{tg} x - \sin x}$$

$$\operatorname{tg} x - \sin x$$

NUMERATORE

$$x \rightarrow 0^+ \rightarrow \sqrt{x} \rightarrow 0$$

$$\cos x = \text{polinomio di grado 4} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$\cos \sqrt{x} = 1 - \frac{1}{2}(\sqrt{x})^2 + \frac{1}{24}(\sqrt{x})^4 + o(\sqrt{x})^4$$

$$= 1 - \frac{1}{2}x + \frac{1}{24}x^2 + o(x^2)$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

N<sub>1</sub>

$$\underbrace{\cos \sqrt{x}} - \underbrace{e^x} + \frac{1}{2} x^2 =$$

$$= \underbrace{1 - \frac{1}{2} x + \frac{1}{2} x^2 + o(x^2)} - \underbrace{\left( 1 + x + \frac{x^2}{2} + o(x^2) \right)}$$

$$+ \frac{1}{2} x^2 = \cancel{1} - \frac{1}{2} x + \frac{1}{2} x^2 + o(x^2) - \cancel{1} - x - \frac{1}{2} x^2 +$$

$$\underbrace{+ o(x^2)} + \cancel{\frac{1}{2} x^2} =$$

$$= -\frac{3}{2} x + \frac{1}{2} x^2 + o(x^2) = x \left[ -\frac{3}{2} + \frac{1}{2} x + o(x) \right]$$

$$D: \quad \text{tg } x - \sin x$$

pol. di grado 3

$$\text{tg } x = 0 + x + \frac{1}{3!} 2x^3 + o(x^3)$$

$$= x + \frac{1}{6} 2x^3 + o(x^3)$$

$$= x + \frac{1}{3} x^3 + o(x^3)$$

$$\sin x = x - \frac{1}{6} x^3 + o(x^3)$$

$$\text{tg } x - \sin x = x + \frac{1}{3} x^3 + o(x^3) - \left( x - \frac{1}{6} x^3 + o(x^3) \right) =$$

$$= \cancel{x} + \frac{1}{3} x^3 + o(x^3) - \cancel{x} + \frac{1}{6} x^3 + o(x^3) = \frac{1}{2} x^3 + o(x^3) = x^3 \left( \frac{1}{2} + o(1) \right)$$

$$f(x) = \text{tg } x$$

$$f'(x) = 1 + (\text{tg } x)^2$$

$$f''(x) = 2 \text{tg } x (1 + (\text{tg } x)^2)$$

$$f^{(3)}(x) = 2(1 + (\text{tg } x)^2)(1 + (\text{tg } x)^2) + 2 \text{tg } x \cdot 2 \text{tg } x (1 + (\text{tg } x)^2)$$

$$\lim_{x \rightarrow 0^+} \frac{x \left( -\frac{3}{2} + \frac{1}{2x} x + o(x) \right)}{x^3 \left( \frac{1}{2} + o(1) \right)} = \frac{+\infty \cdot \left( -\frac{3}{2} \right)}{\frac{1}{2}} = -\infty$$

*Note: The original image contains a red circle around the numerator and a blue circle around the denominator. A blue arrow points from the blue circle to the expression  $-\frac{3}{2} + 0 + 0$ . A red arrow points from the red circle to the expression  $\frac{1}{x^2} \rightarrow +\infty$ .*

$$\frac{1}{x^2} \rightarrow +\infty$$