Local version Morrey

Let $U \subseteq \mathbb{R}^n$ open bdd of class C^1, $p > n$

$W^{1,p}(U) \hookrightarrow C^{0, 1-\frac{n}{p}}(U)$. CONTINUOUS EMB.

$W^{1,p}(U) \hookrightarrow C(\overline{U}), \|\cdot\|_\infty$; $C^{0,\alpha}(U)$ COMPACT EMBEDDING $(\alpha < 1 - \frac{n}{p})$.

In particular $\exists\, C = C(n,p,U)$ $\|f\|_\infty \leq C \|f\|_{W^{1,p}(U)}$

$$|f(x) - f(y)| \leq C \cdot \|f\|_{W^{1,p}} |x-y|^{1-\frac{n}{p}}$$

Proof : $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ extension.

Apply Morrey to $E(f)$ and then restrict to U.

(recall $E(f)$ has compact support..)

GENERALIZED RADEMACHER THEOREM

(Rademacher theorem is only on locally lipschitz f.)

Let $U \subseteq \mathbb{R}^n$ open set, $p \in (n, +\infty]$

Let $f \in W^{1,p}_{loc}(U)$. Then f is differentiable almost everywhere and the a.e. derivatives of f coincide with the weak derivatives of f.

Def f differentiable at $x \in U$ if $\exists\, a \in \mathbb{R}^n$ such that $\lim\limits_{y \to x} \dfrac{|f(y) - f(x) - a \cdot (y-x)|}{|y-x|} = 0$

$a = \left(\dfrac{\partial f}{\partial x_1}(x), \ldots, \dfrac{\partial f}{\partial x_n}(x) \right)$

Def $f \in W^{1,p}_{loc}(U)$ if $\forall\, V \subset\subset U$ $f \in W^{1,p}(V)$.

Obs $f \in W^{1,\infty}_{loc}(U) \to f \in W^{1,\infty}(V)$ $\forall V \subset\subset U$ $\Longrightarrow$ $f \in W^{1,p}(V)$ $\forall V \subset\subset U$

since V is bdd

$\Longrightarrow f \in W^{1,p}_{loc}(U)$.

So it is sufficient to prove the theorem for $W^{1,p}_{loc}(U)$, $p < +\infty$.

Obs Not all Hölder continuous functions satisfies the Rademacher theorem.

Ex f Cantor function is $\frac{\log 2}{\log 3}$ Hölder, but its weak derivative DOES NOT EXIST.

Ex trajectories of the Brownian motion are $\frac{1}{3}$-Hölder continuous curves NOWHERE DIFFERENTIABLE

Proof Fix $x \in U$. And consider the weak derivatives of f $\frac{\partial f}{\partial x_i}(\cdot) \in L^p_{loc}(U)$. Fix a representative $\frac{\partial f}{\partial x_i}(\cdot)$

Define $v(y) = f(y) - f(x) - \sum_i \frac{\partial f}{\partial x_i}(x) \cdot (y_i - x_i) =$

$$= f(y) - f(x) - \underbrace{Df(x)}_{\to \text{weak gradient.}} \cdot (y-x)$$

if we prove that for every x in $U \setminus A$, for $|A|=0$, it holds
$$\lim_{y \to x} \frac{|v(y) - v(x)|}{|y-x|} = 0$$
we get 1) f differentiable at x
2) $\frac{\partial f}{\partial x_i}(x)$ coincides with classical derivatives at x

From the proof of Morrey:

$\forall y \in U$ with $|y-x| = r$

$$|f(x) - f(y)| \leq \frac{2^n}{\omega_n r^n} \left[\int_{B(x,2r)} |f(x) - f(z)|\, dz + \int_{B(y,r)} |f(y) - f(z)|\, dz \right] \leq$$

$$\leq \overline{C}(n,p) \left[\int_{B(x,2r)} |Df|^p \right]^{\frac{1}{p}} \cdot r^{1-\frac{n}{p}} = \overline{C}\, \|Df\|_{L^p(B(x,2r))} \cdot |x-y|^{1-\frac{n}{p}}$$

Apply the previous $\boxed{\text{NOT to } f \text{ but to } v.}$

$$|v(y)-v(x)| \le \bar{C}\, \| |Dv| \|_{L^p(B(x,2r))}\, r^{1-\frac{n}{p}} = \bar{C}\left[\int_{B(x,2r)} |Dv(y)|^p\,dy\right]^{\frac{1}{p}} r^{1-\frac{n}{p}}$$

$$= \bar{C}\left[\int_{B(x,2r)} |Df(y)-Df(x)|^p\,dy\right]^{\frac{1}{p}} r^{-n/p}\, \underbrace{|x-y|}_{=\,r} =$$

$$\boxed{\begin{aligned}Dv(y)&=\\ Df(y)&-Df(x)\end{aligned}}$$

w

$$= \tilde{C}\left[\frac{1}{\omega_n 2^n r^n}\int_{B(x,2r)} |Df(y)-Df(x)|^p\,dy\right]^{1/p} |x-y|$$

$$\frac{|v(y)-v(x)|}{|y-x|} \le \tilde{C}\left[\frac{1}{|B(x,2r)|}\int_{B(x,2r)} |Df(y)-Df(x)|^p\,dy\right]^{1/p}$$

$$\lim_{\substack{y\to x\\ |y-x|=r}} \frac{|v(y)-v(x)|}{|y-x|} \le \lim_{r\to 0}\, \underbrace{\tilde{C}\cdot\left[\frac{1}{|B(x,2r)|}\int_{B(x,2r)} |Df(y)-Df(x)|^p\right]^{1/p}}_{} = 0$$

for a.e x

by Lebesgue theorem

$$h(y) = |Df(y)-Df(x)|^p \in L^1_{loc}(\mathbb{R}^n)$$

for a.e x $\displaystyle\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} h(y) = h(x).$

Theorem (CHARACTERIZATION of LIPSCHITZ CONT.)

$U \subseteq \mathbb{R}^n$ open set

$$f \in W^{1,\infty}_{loc}(U) \iff f \text{ is locally Lipschitz in } U$$

(that is $\forall\, V \subset\subset U$ $f \in W^{1,\infty}(V)$)

($\forall\, V \subset\subset U$ $\exists\, C_V > 0$ $|f(x) - f(y)| \le C_V |x-y|$ $\forall\, x, y \in \overline{V}$.)

Obs 1 No corresponding characterization for Hölder continuous functions!

Obs 2 As a direct by-product (there exists also a direct proof of it ...)

$\boxed{\text{RADEMACHER THEOREM}}$: every locally Lipschitz function is differentiable a.e (and its weak derivatives coincide with the a.e derivatives).

proof Assume $f \in W^{1,\infty}_{loc}(U)$ and prove f locally Lipschitz

$\boxed{n=1}$ we prove that $\forall (a,b)$ open bdd interval

$f \in W^{1,\infty}(a,b) \longrightarrow f$ is absolutely continuous

and $f(y) = f(x) + \int_y^x f'(t)\,dt$

where $f'(t)$ is the weak derivative which coincide with the derivative a.e.

$|f(y) - f(x)| \leq \|f'\|_\infty \cdot |x-y|$

$\forall y, x \in [a,b]$.

$\boxed{n>1}$ $f \in W^{1,\infty}_{loc}(U) \rightarrow \forall x \in U \; \exists r \; B(x,r) \subset\subset U \Rightarrow f \in W^{1,\infty}(B(x,r))$

$\Rightarrow f \in W^{1,p}(B(x,r))$ for $p < +\infty \Rightarrow$ (Morrey) $\Rightarrow f \in C(\overline{B(x,r)})$.

So $f \in C(U)$

Let $\varepsilon > 0$, $U_\varepsilon = \{x \in U \;\; |x| < \frac{1}{\varepsilon},\; \mathrm{dist}(x, \mathbb{R}^n \setminus U) > \varepsilon\}$

$U = \bigcup_{\varepsilon>0} U_\varepsilon$. $\boxed{U_\varepsilon \text{ are bdd}}$

Since $\overline{U_\varepsilon}$ is compact then from the open cover

$\bigcup_{x \in U_\varepsilon} B(x,r) \supseteq \overline{U_\varepsilon}$ we extract a finite cover

where $x_i \in U_\varepsilon$ $r_i > 0$

$\overline{U_\varepsilon} \subseteq \bigcup_{i=1}^{m} B(x_i, r_i) := V_\varepsilon$ $V_\varepsilon \subset\subset U$ $\mathrm{dist}(V_\varepsilon, \mathbb{R}^n \setminus U) > \varepsilon - \max_{i=1..m} r_i$

$\boxed{\delta < \varepsilon - \max_i r_i}$ and $\eta_\delta(y) = \frac{1}{\delta^n} \eta\left(\frac{y}{\delta}\right) \in C_c^\infty(\mathbb{R}^n)$ ~~standard~~ mollifier

$\mathrm{supp}\, \eta_\delta(y) = B(0,\delta)$ $\eta_\delta \geq 0$ $\|\eta_\delta\|_{L^1} = 1$

$f * \eta_\delta \in C^\infty(V_\varepsilon)$ and moreover (since $f \in C(U)$)

$f * \eta_\delta \to f$ uniformly in V_ε as $\delta \to 0$

take $x, y \in B(x_i, r_i)$ then $x + t(y-x) \in B(x_i, r_i)$ $\forall t \in [0,1]$

$$|f * \eta_\delta(y) - f * \eta_\delta(x)| = \left| \int_0^1 \frac{d}{dt} f * \eta_\delta(x + t(y-x))\, dt \right| = \int_0^1 |D(f * \eta_\delta)(x + t(y-x))$$

$$\leq \int_0^1 |D(f * \eta_\delta)(x + t(y-x))|\, |x-y|\, dt \leq \|D(f * \eta_\delta)\|_{L^\infty(B(x_i, r_i))} |x-y|$$

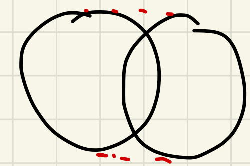
$$\Longrightarrow \forall x,y \in B(x_i,r_i) \quad |f*\eta_\delta(x) - f*\eta_\delta(y)| \le \| Df\|_{L^\infty(B(x_i,r_i))} |x-y|$$

$$\delta \to 0 \quad \downarrow$$

$$\boxed{\forall x,y\in\overline{B(x_i,r_i)}} \quad \boxed{\begin{aligned}|f(x)-f(y)| &\le \|Df\|_{L^\infty(B(x,r_i))}|x-y| \\ &\le \|Df\|_{L^\infty(V_\varepsilon)}|x-y|\end{aligned}}$$

Now.

$$\overline{V_\varepsilon} = \bigcup_{i=1}^{M} \overline{B(x_i,r_i)}$$



Let $x\in \overline{B(x_i,r_i)}\setminus\overline{B(x_j,r_j)}$ $\quad y\in\overline{B(x_i,r_i)\setminus B(x_j,r_i)}$

1) if $\overline{B(x_i,r_i)}\cap\overline{B(x_j,r_j)}\ne\emptyset$ or $C_{ij}\le 1$ $\quad |x-y|\ge C_{ij}\,[|x-z|+|z-y|]$

$$z\in \overline{B(x_i,r_i)}\cap\overline{B(x_j,r_j)}$$

$$\Longrightarrow |f(x)-f(y)|\le |f(x)-f(z)|+|f(z)-f(x)|\le$$

$$\le \cancel{2}\,\|Df\|_{L^\infty(V_\varepsilon)}\,[|x-z|+|z-y|]\le \frac{\|Df\|_{L^\infty(V_\varepsilon)}}{C_{ij}}|x-y|$$

2) if $\overline{B(x_i,r_i)} \cap \overline{B(x_j,r_j)} = \emptyset$ $\quad \delta_{ij} = \mathrm{dist}(\overline{B(x_i,r_i)}, \overline{B(x_j,r_j)})$

$$|f(x)-f(y)| \leq \frac{2\|f\|_{L^\infty(V_\varepsilon)}}{\delta_{ij}} \delta_{ij} \leq \frac{2\|f\|_{L^\infty(V_\varepsilon)}}{\delta_{ij}} |x-y|$$

$\Rightarrow$ f is lipschitz in V_ε $\Rightarrow$ f is lipschitz in U_ε

$\Rightarrow$ f is locally lipschitz in U.

($\forall\, V \subset\subset U$ $\exists\, \varepsilon_0$ such that $V \subseteq U_{\varepsilon_0}$)

NOTE $f \in W^{1,\infty}(U) \not\Rightarrow f$ Lipschitz in U ! (just locally lipschitz)

LOOKING AT THE PROOF:

If we have $f \in W^{1,\infty}(U)$ $\quad U$ convex, by the previous proof

$$\Rightarrow \quad |f(x)-f(y)| \leq \|Df\|_{L^\infty(U)} |x-y|$$

(so f is lipschitz in U).

in particular $f \in W^{1,\infty}(\mathbb{R}^n) \Rightarrow |f(x)-f(y)| \le \|Df\|_\infty |x-y|$
$\forall x, y \in \mathbb{R}^n$

(f also bounded)

If U is CONNECTED (so in $\mathbb{R}^n$ equivalent to PATHWISE CONNECTED)

and $f \in W^{1,\infty}(U)$ the same proof gives:

$$\forall x, y \in U \qquad |f(x)-f(y)| \le \|Df\|_{L^\infty(U)} \cdot d_U(x,y)$$

where $d_U(x,y) =$ geodesic distance in U =

= infimum of the lenght of rectifiable curves in U joining x and y.

$d_U(x,y)$ in general can be ~~very~~ different of $|x-y|$!