U open bdd of class C^1, $p<n$

$\forall q \in [1, p^*]$ $\exists\, C(n,p,q,U)$ such that $\forall f \in W^{1,p}(U)$

$$\|f\|_{L^q(U)} \leq C\, \|f\|_{W^{1,p}(U)}$$

proof $E: W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ (for fixed $V \supset\supset U$)

Apply (GNS) to Ef $\Rightarrow$ $\|Ef\|_{L^{p^*}} \leq C(n,p)\, \|Ef\|_{W^{1,p}(\mathbb{R}^n)}$

$$\leq \tilde{C}(n,p,U)\, \|f\|_{W^{1,p}(U)}$$

$$\|f\|_{L^{p^*}(U)} \leq \|Ef\|_{L^{p^*}(\mathbb{R}^n)} \leq \tilde{C}(n,p,U)\, \|f\|_{W^{1,p}(U)}$$

U bdd $\underset{\text{Holder}}{\Longrightarrow}$ $\|f\|_{L^q(U)} \leq |U|^{\frac{p^* q}{p^*-q}}\, \|f\|_{L^{p^*}(U)} \leq C(n,p,U,q)\, \|f\|_{W^{1,p}(U)}$.

Obs $f \in W^{1,n}(U)$, U bdd $\Rightarrow$ $f \in W^{1,p}(U)$ $\forall p<n$ $\Rightarrow$

$$\|f\|_{L^q(U)} \leq \tilde{C}(n,p,q,U)\, \|f\|_{W^{1,n}(U)} \quad \forall q \in [1,+\infty).$$

$\boxed{\text{Case } p\in(n,+\infty)}$

MORREY INEQUALITY.

$\boxed{\text{Let } n<p<+\infty}$ Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ continuously

$$\left(\|f\|_\infty \le C(n,p)\, \|f\|_{W^{1,p}(\mathbb{R}^n)} \right)$$

Moreover $\exists\, C=C(n,p)$ such that

$$|f(x)-f(y)| \le C\,|x-y|^{1-\frac{n}{p}}\, \|\,|Df|\,\|_{L^p(\mathbb{R}^n)} \quad \text{for a.e } x,y$$

f is bdd and Hölder continuous of exp: $1-\frac{n}{p}$, UP TO CHOOSING A REPR.

(NOT VICEVERSA obviously! since bdd Hölder functions are in general not $W^{1,p}(\mathbb{R}^n)$ – differently from $p=+\infty$ as we will see)

Proof First observation: if $\exists\, C(n,p)$ $\|f\|_\infty \le C(n,p)\,\|f\|_{W^{1,p}}$

$|f(x)-f(y)| \le C\,|x-y|^{1-\frac{n}{p}}\,\|\,|Df|\,\|_{L^p}$

holds for all $f\in C^\infty_c(\mathbb{R}^n)$, then it holds $\underline{\forall\, f\in W^{1,p}(\mathbb{R}^n)}$.

Indeed

$\forall$ ~~Let~~ $f\in W^{1,p}(\mathbb{R}^n) \to \exists\, \phi_k \in C^\infty_c(\mathbb{R}^n)$ $\phi_k \to f$ in $W^{1,p}(\mathbb{R}^n)$

$\Longrightarrow \phi_k$ is Cauchy in $\|\cdot\|_{1,p} \Longrightarrow$

ϕ_k is Cauchy in $\|\cdot\|_\infty$ and ϕ_k are equi-Hölder.
For Ascoli-Arzelà up to subsequence $\phi_k \to \tilde{f}$ locally uniformly. $\tilde{f}$ is continuous in $\mathbb{R}^n$, $\tilde{f} = f$ a.e. so f has a continuous representative. Since the limit is the same for all subsequence ϕ_k, the entire sequence is converging to f

$\phi_k \to f$ also pointwise $\Longrightarrow$ $|\phi_k(x) - \phi_k(y)| \leq C|x-y|^{1-\frac{n}{p}} \||Df|\|_{W^{1,p}}$

$$\Longrightarrow |f(x) - f(y)| \leq C|x-y|^{1-n/p} \||Df|\|_{L^p} \qquad x \neq y.$$

Moreover $\phi_k \in (L^\infty(\mathbb{R}^n), \|\cdot\|_\infty)$ is also Cauchy $\Rightarrow \phi_k \to f$ in L^∞

$\Rightarrow f \in L^\infty(\mathbb{R}^n) \qquad \|f\|_{L^\infty(\mathbb{R}^n)} \leq C(n,p)\|f\|_{W^{1,p}(\mathbb{R}^n)}$

$\boxed{\text{Note that } \phi_k \to f \text{ UNIFORMLY in } \mathbb{R}^n}$ by Morrey inequality.

So we are reduced to $f \in C_c^\infty(\mathbb{R}^n)$

Let $f \in C_c^\infty(\mathbb{R}^n)$.

Claim: $\exists\, C = C(n)$ $\forall x, \forall r$

$$\underbrace{\frac{1}{\omega_n r^n}\int_{B(x,r)} |f(x)-f(y)|}_{} \le C \int_{B(x,r)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy$$

$$\int_{B(x,r)} |f(x)-f(y)|\,dy = \int_0^r \underbrace{\int_{\partial B(0,1)} |f(x)-f(x+sw)|\, s^{n-1}\, d\mathcal{H}^{n-1}(w)}_{}\, ds$$ by int on spheres.

$|w|=1$!

$$|f(x)-f(x+sw)| = \left|\int_0^s \frac{d}{dt} f(x+tw)\,dt\right| \le \int_0^s |Df(x+tw)|\cdot|w|\,dt = \int_0^s |Df(x+tw)|\,dt$$

$$\int_{\partial B(0,1)} |f(x)-f(x+sw)|\, s^{n-1}\, d\mathcal{H}^{n-1}(w) \le \int_{\partial B(0,1)} \int_0^s |Df(x+tw)|\,dt\ s^{n-1}\, d\mathcal{H}^{n-1}(w) =$$

$$= \text{change order} = \int_0^s \underbrace{\int_{\partial B(0,1)} |Df(x+tw)|\, d\mathcal{H}^{n-1}(w)}_{}\ s^{n-1}\,dt =$$

$$= s^{n-1}\int_0^s \int_{\partial B(x,t)} \frac{|Df(y)|}{t^{n-1}}\, d\mathcal{H}^{n-1}(y) = \text{NB } t = |x-y| = s^{n-1}\int_0^s \int_{\partial B(x,t)} \frac{|Df(y)|}{|x-y|^{n-1}}\, d\mathcal{H}^{n-1}\, dt$$

$$= s^{n-1}\int_{B(x,s)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy \le s^{n-1}\int_{B(x,r)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy.$$

$$\underline{\int_{B(x,r)} |f(y)-f(x)|} = \int_0^r \int_{\partial B(0,1)} s^{n-1} |f(x)-f(x+sw)|\, d\mathcal{H}^{n-1}(w) \le \underbrace{\int_0^r s^{n-1}}_{\frac{r^n}{n}} \int_{B(x,r)} \frac{|Df(y)|}{|x-y|^{n-1}}\, dy$$

CLAIM PROVED with $C = \frac{1}{n\omega_n}$.

Holder

$$\frac{1}{r^n\omega_n} \int_{B(x,r)} |f(y)-f(x)| \le \frac{1}{n\omega_n} \| |Df| \|_{L^p} \cdot \underbrace{\left[\int_{B(x,r)} \left(\frac{1}{|x-y|^{n-1}} \right)^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}}}$$

OBSERVE $(n-1)\frac{p}{p-1} < n \iff p > n$

$$\left[\int_{B(x,r)} \left(\frac{1}{|x-y|^{n-1}} \right)^{\frac{p}{p-1}} dy \right]^{\frac{p-1}{p}} = \left[\int_0^r \frac{s^{n-1}}{s^{(n-1)\frac{p}{p-1}}}\, n\omega_n\, ds \right]^{\frac{p-1}{p}} = (n\omega_n)^{\frac{p-1}{p}} \cdot \left[\int_0^r s^{-\frac{n-1}{p-1}}\, ds \right]^{\frac{p-1}{p}} =$$

$$= (n\omega_n)^{\frac{p-1}{p}} \left[\left(\frac{p-1}{p-n} \right) \cdot r^{\frac{p-n}{p-1}} \right]^{\frac{p-1}{p}} = \left[(n\omega_n) \frac{p-1}{p-n} \right]^{\frac{p-1}{p}} \cdot r^{\frac{p-n}{p}} = C(n,p)\, r^{1-\frac{n}{p}}$$

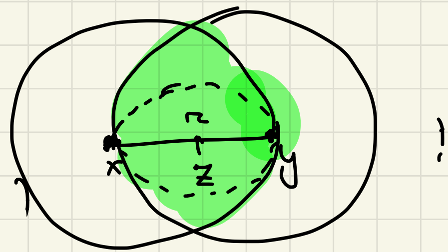
$$\frac{1}{\omega_n r^n} \int_{B(x,r)} |f(y)-f(x)| \le C(n,p)\, r^{1-\frac{n}{p}} \| |Df| \|_{L^p} \quad (*)$$

Apply ⊛ $r=1$ fix $x\in\mathbb{R}^n$

$$|f(x)| \le \frac{1}{\omega_n}\int_{B(x,1)} |f(x)-f(y)| + \underbrace{\frac{1}{\omega_n}\int_{B(x,1)} |f(y)|\,dy}_{} \le C(n,p)\,\||Df|\|_{L^p} + \frac{C}{\omega_n}\|f\|_{L^p}$$

Holder.

$$\Rightarrow \|f\|_\infty \le \bar{C}(n,p)\,\|f\|_{W^{1,p}}.$$



!

Apply ⊛ x,y $|x-y|=r$

$B(x,r)\cap B(y,r)=W$

$z=\frac{x+y}{2}$ $\quad t\in B(z,\frac{r}{2}) \Rightarrow |t-x|\le|t-z|+|z-x|<\frac{r}{2}+\frac{r}{2}=r$

$B(z,\frac{r}{2})\subseteq W=B(x,r)\cap B(y,r) \Rightarrow |W|\ge \omega_n\frac{r^n}{2^n}$.

$$|f(x)-f(y)| = \frac{1}{|W|}\int_W |f(x)-f(y)|\,dz \le \frac{1}{|W|}\int_W |f(x)-f(z)|\,dz + \frac{1}{|W|}\int_W |f(z)-f(y)|\,dz$$

$$\le \frac{2^n}{\omega_n r^n}\int_{B(x,r)} |f(x)-f(z)|\,dz + \frac{2^n}{\omega_n r^n}\int_{B(y,r)} |f(y)-f(z)|\,dz \le$$

$$\le 2^{n+1}\,C(n,p)\,\||Df|\|_{L^p(\mathbb{R}^n)}\, r^{1-\frac{n}{p}} = \bar{C}(n,p)\,\||Df|\|_{L^p}\,|x-y|^{1-\frac{n}{p}}.$$

REMARK $f \in W^{1,p}(\mathbb{R}^n)$ $p > n$. Then

f is the UNIFORM limit of $\phi_k \in C_c^\infty(\mathbb{R}^n)$

Take $\phi_k \to f$ in $W^{1,p}(\mathbb{R}^n)$ $\phi_k \in C_c^\infty(\mathbb{R}^n)$

$\Rightarrow \|\phi_k - f\|_\infty \leq C(n,p) \|\phi_k - f\|_{W^{1,p}(\mathbb{R}^n)}$

$\Rightarrow \boxed{\phi_k \to f \text{ UNIFORMLY}}$

therefore $\lim_{|x| \to +\infty} f(x) = 0 \quad \forall f \in W^{1,p}(\mathbb{R}^n)$