$(\Omega, \mathcal{F}, \mathbb{P})$ probability space

X random variable $\quad X: \Omega \to \mathbb{R}$ measurable funct.

$\forall A \in \mathcal{B}(\mathbb{R}) \quad X^{-1}(A) = \{\omega \in \Omega \quad X(\omega) \in A\} \in \mathcal{F}$

X induces on $\mathbb{R}$ a Borel measure $\mathbb{P}_X$ (a Borel on $\mathbb{R}$)

$$\mathbb{P}_X(A) = \mathbb{P}\{\omega \in \Omega \mid X(\omega) \in A\} = \mathbb{P}(X^{-1}(A))$$

$\mathbb{P}_X$ is a FINITE BOREL MEASURE $\quad \left(\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1 \right)$

$\updownarrow$

it is associated to a cumulative distribution function

$$G_X(x) = \mathbb{P}_X(-\infty, x] = \mathbb{P}\{\omega \mid X(\omega) \leq x\}$$

$\downarrow$

right continuous $\quad \lim_{x \to a^+} G(x) = G(a)$, NON DECREASING we may define the notion of integral with respect to any μ Borel σ-finite or finite measure

$f : \mathbb{R} \to \mathbb{R}$ continuous,

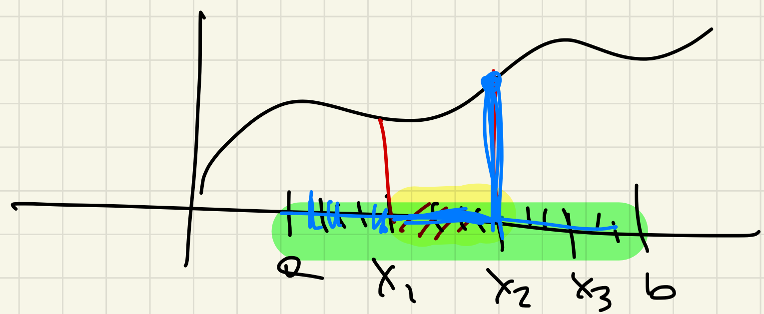
$$\underbrace{\int_{[a,b]} f(x)\, d\mathbb{P}_X(x)} = \sup\left[\sum_{n=1}^{k+1} f(x_n)\left[G_X(x_n) - G_X(x_{n-1})\right] \right],$$

$\left.\begin{array}{l}\text{sup among all possible subdivision} \\ x_0 = a < x_1 < \dots < x_{n+1} = b\end{array}\right\}$

take a subdivision of (a,b) $\underbrace{x_0 = a < x_1 < x_2 < \dots < x_k < x_{k+1} = b}$

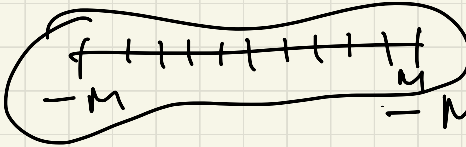
$$\sum_{n=1}^{k+1} \underbrace{f(x_n)} \cdot \underbrace{\mathbb{P}_X(x_{n-1}, x_n]}$$

$$= \sum_{n=1}^{k+1} f(x_n) \cdot \left[G_X(x_n) - G_X(x_{n-1})\right]$$



$$\int_{\mathbb{R}} f(x)\, d\mathbb{P}_X(x) = \sup_{(-M,M]} \sup \left\{ \sum_{k=1}^{n+1} f(x_n)\,[G_X(x_n) - G_X(x_{n-1})], \right.$$

among all possible



$$\left. -M = x_0 < x_1 < x_2 < \dots < x_{k+1} = M \right\}$$

$\forall\, i \neq j \quad |x_i - x_{i-1}| = |x_j - x_{j-1}|$

If X is an absolutely continuous random variable

$\underline{G'_X(x) = g(x)}$ for a.e. x

$\underline{g(x) \geq 0}$ density

$$\int_{\mathbb{R}} f(x)\, d\mathbb{P}_X(x) = \int_{\mathbb{R}} f(x)\, \underline{g(x)\, dx}$$

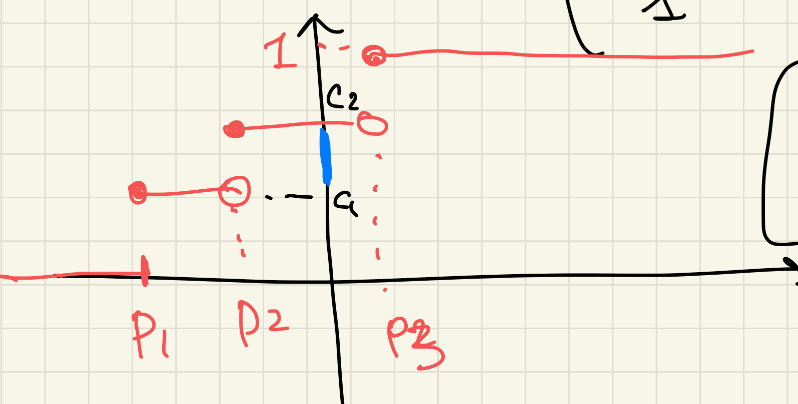
If X is a discrete random variable

↓ X takes values only on a discrete set $\{p_1 \dots p_n\}$

$$G_X(x) = \begin{cases} 0 = c_0 & x < p_1 \\ c_1 & p_1 \le x < p_2 \\ c_2 & p_2 \le x < p_3 \\ c_3 & \\ \vdots & \\ 1 & p_n \le x < +\infty \end{cases}$$

$0 < c_1 < c_2 < \dots < 1$

$$P_X(x) = \sum_{i=1}^{n} (c_i - c_{i-1})\, \delta_{\{p_i\}}$$



$$\boxed{\int_{\mathbb{R}} f(x)\, dP_X(x) = \sum_{i=1}^{n} f(p_i)\cdot (c_i - c_{i-1})}$$

e.g. $f(x) = x^k \quad k \geq 1$ cont.

$$\int_{\mathbb{R}} x^k \, d\mathbb{P}_X(x) \underset{\substack{\uparrow \\ \text{if it exists}}}{=} \underbrace{\mathbb{E}(X^k)}$$

mean of X random variable $\boxed{\mathbb{E}(X) := \int_{\mathbb{R}} x \, d\mathbb{P}_X(x)}$

k-moment of X random variable is

$$\underbrace{\mathbb{E}(|X|^k) := \int_{\mathbb{R}} |x|^k \, d\mathbb{P}_X(x)}$$

$$\boxed{\mathbb{E}(X^k) < +\infty \iff \mathbb{E}(|X|^k) < +\infty}$$

It is not completely obvious but it is true that if

$$\underbrace{\int_{\mathbb{R}} x^k \, dP_X(x) < +\infty}$$

then

$$\int_{\mathbb{R}} |x|^k \, dP_X(x) < +\infty$$

and also the viceversa is true.

M = space of all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$

$\downarrow$ $X \; \cancel{\omega} : \Omega \to \mathbb{R}$ meas.

$$M_k = \left\{ X \in M, \; X \text{ random variable such that } \int_{\mathbb{R}} |x|^k \, dP_X(x) < +\infty \right\}$$

$M_1 = \{$random variables which have finite mean$\}$

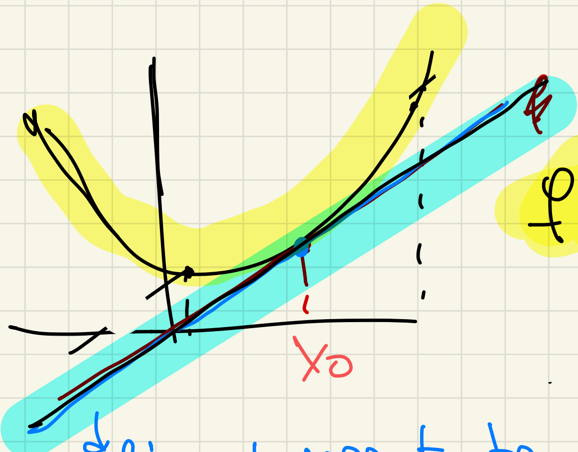
$M_2 = \{$random variables which have finite $E(X^2) = \int_{\mathbb{R}} x^2 \, dP_X(x) < +\infty \}$

"Jensen inequality": if $f(x): \mathbb{R} \to \mathbb{R}$ convex (smooth and convex $f''(x) \geq 0 \ \forall x$)

$$\int_{\mathbb{R}} f(x) \, dP_X(x) \geq f\left[\int_{\mathbb{R}} x \, dP_X(x) \right]$$

$\forall X$ random variable

proof. we have to use that $\int_{\mathbb{R}} dP_X(x) = 1$.



f convex and smooth.

$$f(x) \geq \boxed{f(x_0) + f'(x_0)\cdot(x - x_0)}$$

line tangent to the graph of f passing in the point $(x_0, f(x_0))$ has equation $y = f(x_0) + f'(x_0)(x-x_0)$

$$x_0 = \int_{\mathbb{R}} z\, dP_X(z) \qquad \forall x$$

$$f(x) \geq f\left(\int_{\mathbb{R}} z\, dP_X(z)\right) + f'\left(\int_{\mathbb{R}} z\, dP_X(z)\right)\left[x - \int_{\mathbb{R}} z\, dP_X(z)\right]$$

$$\int_{\mathbb{R}} f(x)\, d\mathbb{P}_X(x) \geq \int_{\mathbb{R}} \underbrace{f\left(\int_{\mathbb{R}} z\, d\mathbb{P}_X(z)\right)}_{\text{CONST}} d\mathbb{P}_X(x) + \int_{\mathbb{R}} f'\left(\int_{\mathbb{R}} z\, d\mathbb{P}_X(z)\right)\left(x - \int_{\mathbb{R}} z\, d\mathbb{P}_X(z)\right) d\mathbb{P}_X(x)$$

$$\int_{\mathbb{R}} f(x)\, d\mathbb{P}_X(x) \geq f\left(\int_{\mathbb{R}} z\, d\mathbb{P}_X(z)\right) \underbrace{\int_{\mathbb{R}} d\mathbb{P}_X(x)}_{=1} +$$

$$+\, f'\left(\int_{\mathbb{R}} z\, d\mathbb{P}_X(z)\right) \cancel{\left[\underbrace{\int_{\mathbb{R}} x\, d\mathbb{P}_X(x) - \underbrace{\int_{\mathbb{R}} 1\, d\mathbb{P}_X(x)}_{1} \cdot \int_{\mathbb{R}} z\, d\mathbb{P}_X(z)}_{=0}\right]}$$

Take $f(x) = |x|^k$ $\quad k > 1$ $\quad$ f convex

$$\int_{\mathbb{R}} f(x)\, d\mathbb{P}_X(x) \geq f\left(\int_{\mathbb{R}} x\, d\mathbb{P}_X(x)\right)$$

$$+\infty > \boxed{\int_{\mathbb{R}} |x|^k\, d\mathbb{P}_X(x) \geq \left|\int_{\mathbb{R}} x\, d\mathbb{P}_X(x)\right|^k}$$

$$\text{if} \quad \int_{\mathbb{R}} |x|^k\, d\mathbb{P}_X(x) < +\infty \implies \int_{\mathbb{R}} x\, d\mathbb{P}_X(x) < +\infty$$

$M^k \subset M^1$ (if X has bdd k-moment then it has bdd mean).

CONVEXITY OF f

Jensen $\quad f(x) \geq f(x_0) + f'(x_0)(x - x_0) \quad \forall x$

apply the convexity inequality to $\quad (n \geq 1)$

$$f(x^n) \geq \underbrace{f(x_0)} + f'(x_0)(x^n - x_0)$$

$$x_0 = \int_{\mathbb{R}} z^n \, d\mathbb{P}_X(z) \quad \text{constant}$$

$$\int_{\mathbb{R}} f(x^n) \, \underbrace{d\mathbb{P}_X(x)} \geq f\left(\int_{\mathbb{R}} z^n \, d\mathbb{P}_X(z) \right) +$$

$$+ f'\left(\int_{\mathbb{R}} z^n \, d\mathbb{P}_X(z) \right) \cdot \left[\int_{\mathbb{R}} x^n \, d\mathbb{P}_X(x) - \int_{\mathbb{R}} z^n \, d\mathbb{P}_X(z) \right]$$

$$\boxed{\int_{\mathbb{R}} f(x^n) \, d\mathbb{P}_X(x) \geq f\left(\int_{\mathbb{R}} z^n \, d\mathbb{P}_X(z) \right)}$$

(some argument gives that $\forall h : \mathbb{R} \to \mathbb{R}$

$$\int_{\mathbb{R}} \underline{\underline{f}}(h(x))\, d\mathbb{P}_X(x) \geq \underline{\underline{f}}\left[\underbrace{\int_{\mathbb{R}} h(z)\, d\mathbb{P}_X(z)}\right]$$

$\underbrace{f \text{ convex}}$ if $\boxed{\underbrace{\int_{\mathbb{R}} h(z)\, d\mathbb{P}_X(z) < +\infty}}$

(in the convexity inequality I substitute x with $h(x)$ and x_0 with $\int_{\mathbb{R}} h(z)\, d\mathbb{P}_X(z)$

$1 < \underline{\underline{n < k}}$ $\quad f(x) = |x|^{\frac{k}{n}}$ convex $\quad h(x) = x^n$

$$+\infty > \int_{\mathbb{R}} |x|^k\, d\mathbb{P}_X(x) = \int_{\mathbb{R}} f(h(x))\, d\mathbb{P}_X \geq \underbrace{\left|\int_{\mathbb{R}} x^n\, d\mathbb{P}_X(x)\right|^{\frac{k}{n}}}$$

$$\boxed{\mathbb{E}|X|^k \geq \left|\mathbb{E}(X)^n\right|^{\frac{k}{n}} \geq \underline{\underline{\left[\mathbb{E}|X|^n\right]^{\frac{k}{n}}}} \quad \boxed{\forall k > n}}$$

(DIS. DI JENSEN)

$$\left[\int_{\mathbb{R}} |x|^k \, dP_X(x)\right]^{\frac{1}{k}} \geq \left[\int_{\mathbb{R}} |x|^n \, dP_X(x)\right]^{\frac{1}{n}}$$

$$\left[\mathbb{E}|X|^k\right]^{1/k} \geq \left[\mathbb{E}|X|^n\right]^{1/n} \qquad \boxed{1 \leq n < k}$$

[We want to put some "differential" structure on these spaces (which are infinite dimensional → the "points" are the random variables.)]

X be an (infinite dimensional) VECTORIAL SPACE on $\mathbb{R}$ $\left(x_1, x_2 \in X \quad \lambda x_1 + \mu x_2 \in X \quad \forall \lambda, \mu \in \mathbb{R} \right)$.

X is a METRIC SPACE if $\exists$ a function

d (distance) : $X \times X \longrightarrow [0, +\infty)$

- $d(x_1, x_2) \geq 0 \quad \forall x_1, x_2 \in X$

  $d(x_1, x_2) = 0 \iff x_1 = x_2$

- $d(x_1, x_2) = d(x_2, x_1) \mapsto$ SYMMETRY COND.

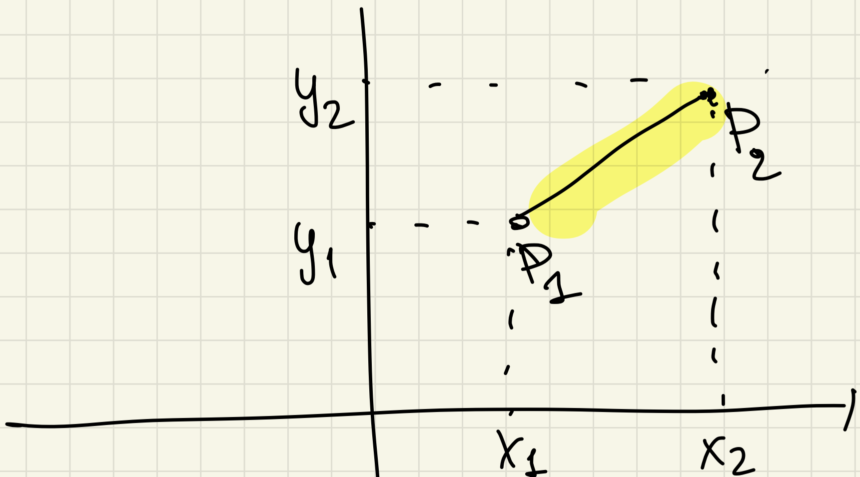
- $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$

  (TRIANGULAR INEQUALITY)

We say that the distance is associated to a NORM if $d(X_1, X_2) = \|X_1 - X_2\|$

$\|\cdot\| : X \to \mathbb{R}$ NORM :

$$\boxed{\begin{array}{l} 1)\ \|x\| \geq 0 \qquad \|x\| = 0 \iff x = 0 \\ 2)\ \|\lambda x\| = |\lambda|\,\|x\| \qquad \forall \lambda \in \mathbb{R}. \\ 3)\ \|x_1 + x_2\| \leq \|x_1\| + \|x_2\|. \end{array}}$$



$\mathbb{R}^2$

distance = length of the segment (P_1, P_2)

$P_1 = (x_1, y_1)$

$P_2 = (x_2, y_2)$

$$d(P_1, P_2) = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$$

$$|P_1 - P_2| =$$

Def X vectorial space on $\mathbb{R}$

is a NORMED SPACE if there exists

a norm $\|\cdot\| : X \to [0, +\infty)$

1) $\|x\| = 0 \iff x = 0$ $\quad \|x\| \geq 0 \ \forall x$

2) $\|\lambda x\| = |\lambda| \ \|x\|$ $\quad \lambda \in \mathbb{R}$ $\qquad \|x_1 - x_2\| = \|x_2 - x_1\|$

3) $\boxed{\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|}$

the norm is associated to a distance

$$d(x_1, x_2) = \|x_1 - x_2\|$$

$y_1 - y_3 = x_1 \qquad x_2 = y_3 - y_2$

$$\|x_1 + x_2\| = \|y_1 - y_3 + y_3 - y_2\| = \|y_1 - y_2\| \leq \underbrace{\|y_1 - y_3\|}_{\|x_1\|} + \underbrace{\|y_3 - y_2\|}_{\|x_2\|}$$

Es $X = \{ f : [0,1] \to \mathbb{R} \text{ CONTINUOUS} \}$

$f_1, f_2 \in X \Rightarrow \lambda f_1 + \mu f_2$ is continuous $: [0,1] \to \mathbb{R}$

$\lambda, \mu \in \mathbb{R}$

NORM (L^∞ NORM)

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$

$\|f\|_\infty \geq 0 \quad \forall f$

$\|f\|_\infty = 0 \iff \max_{[0,1]} |f(x)| = 0$

$\|\lambda f\|_\infty = |\lambda| \, \|f\|_\infty$

$$\boxed{d_\infty(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|}$$

$\|f+g\|_\infty = \max_x |f(x)+g(x)| \leq$

$\leq \max_x [|f(x)| + |g(x)|]$

$\leq \max |f(x)| + \max |g(x)|$

$$\|f\|_1 = \int_0^1 |f(x)|\,dx$$

IT IS also a NORM on X

$$\|f\|_1 \geq 0 \qquad \int_0^1 |f(x)|\,dx = 0 \iff f(x) = 0 \ \forall x$$

$$\|\lambda f\|_1 = \int_0^1 |\lambda f(x)|\,dx = |\lambda| \int_0^1 |f(x)|\,dx = |\lambda|\,\|f\|_1$$

$$\|f+g\|_1 = \int_0^1 |f(x)+g(x)|\,dx \leq \int_0^1 |f(x)|+|g(x)|\,dx \leq$$

$$\leq \int_0^1 |f(x)|\,dx + \int_0^1 |g(x)|\,dx$$

$$\boxed{d_1(f,g) = \int_0^1 |f(x)-g(x)|\,dx}$$

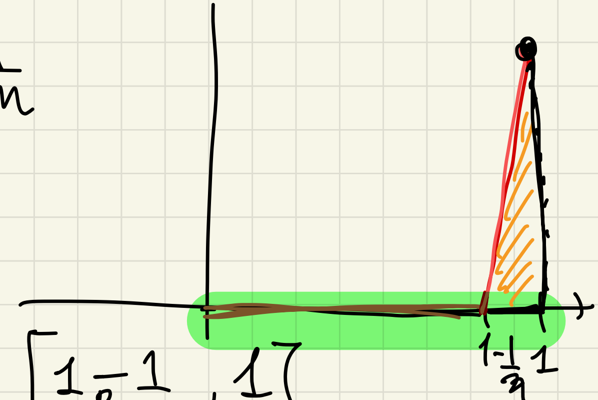
$\underline{\underline{d_\infty(f,g) \le \delta}} \iff |f(x)-g(x)| \le \delta \quad \forall x$

$g(x)-\delta \le f(x) \le g(x)+\delta \quad \forall x \in [0,1]$

$d_1(f,g) \le \delta \iff \int_0^1 |f(x)-g(x)|\,dx \le \delta$

$\int_0^1 g(x)dx - \delta \le \int_0^1 f(x)dx \le \int_0^1 g(x)dx + \delta$

Ex $f_n(x) = \begin{cases} 0 & x < 1 - \frac{1}{n} \\ n\left(x - 1 + \frac{1}{n}\right) \end{cases}$



$g(x) \equiv 0 \quad \forall x \in [0,1]$

$x \in \left[1 - \frac{1}{n}, 1\right]$

$$d_1(f_n, 0) = \int_0^1 |f_n(x) - 0|\, dx =$$

$$= \int_0^1 n\left(x - 1 + \frac{1}{n}\right) dx = \frac{1}{2n} < 1$$

$$d_\infty(f_n, 0) = \max_{x \in [0,1]} |f_n(x) - 0| = 1 \quad \forall n$$

$$\lim_{n \to +\infty} d_1(f_n, 0) = 0$$

$$\lim_{n \to +\infty} d_\infty(f_n, 0) = 1$$

f_n is "going" to the constant function 0 in the sense of the d_1 distance, but not in the sense of the d_∞ distance.

If X is a NORMED SPACE ($X, \|\cdot\|$) and $x_n \in X$ $\forall n$ and $x \in X$ we say that

$$\lim_{n \to +\infty} x_n = x \iff \lim_{n \to +\infty} \|x_n - x\| = 0$$

$x_n \to x$ iff DISTANCE BETWEEN x_n, x GOES TO 0.

<u>Definition</u>, Let X be a vectorial space on $\mathbb{R}$ and a norm $\|\cdot\|$ on it.

We say that the normed space $(X, \|\cdot\|)$ is COMPLETE if

$\forall (x_n)$ sequence in X such that

$$\lim_{n,m \to +\infty} \|x_n - x_m\| = 0$$ (such sequences are called CAUCHY SEQUENCES)

it holds that there exists $x \in X$ such that

$$\lim_n \|x_n - x\| = 0$$

(so Cauchy sequences are converging to some element in the space X).

A normed complete space is called a BANACH SPACE obs

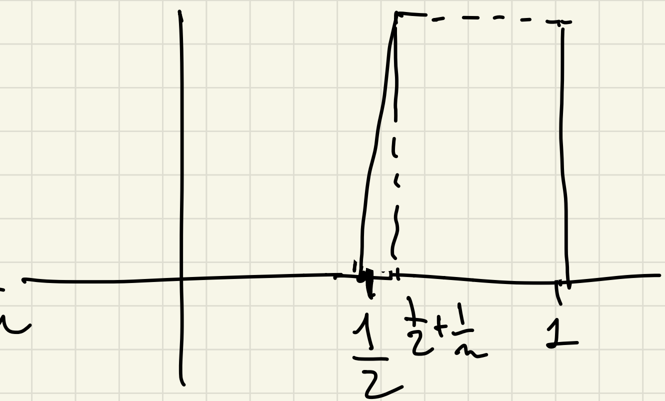
Not all the normed space are complete.

$$X = \{ f : [0,1] \to \mathbb{R} \text{ continuous} \}$$

$$\|f\|_1 = \int_0^1 |f(x)|\, dx$$

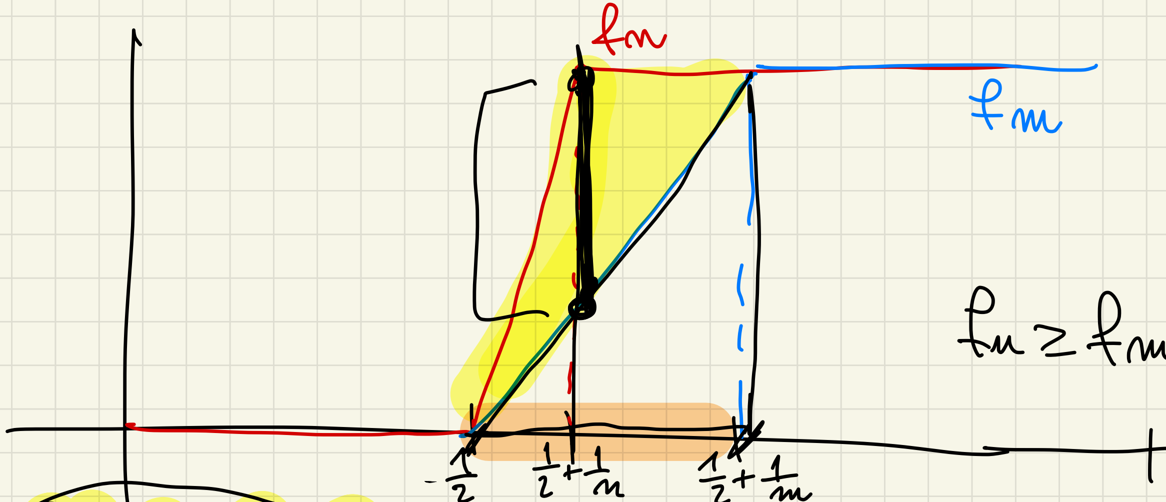
$$\underline{f_n(x)} = \begin{cases} 0 & x \le \frac{1}{2} \\ n\left(x - \frac{1}{2}\right) & \frac{1}{2} < x < \frac{1}{2} + \frac{1}{n} \\ 1 & x \ge \frac{1}{2} + \frac{1}{n} \end{cases}$$

$n \geq 2$



is a Cauchy sequence

$$\int_0^1 |f_n(x) - f_m(x)|\, dx \to 0$$



$n > m$

$\frac{1}{2}+\frac{1}{n} < \frac{1}{2}+\frac{1}{m}$

$f_n \geq f_m$

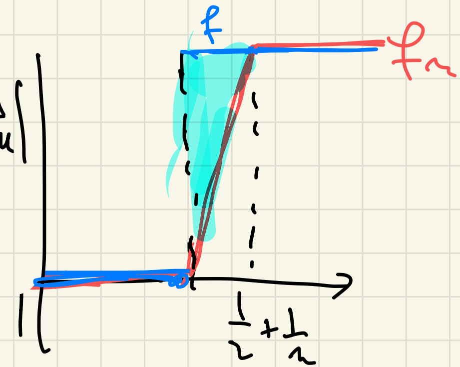
$$\int_0^1 |f_n(x)-f_m(x)|\,dx = \int_{1/2}^{1/2+1/n} n\left(x-\frac{1}{2}\right) - m\left(x-\frac{1}{2}\right) dx \; +$$

$$+ \int_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+\frac{1}{m}} 1 - m\left(x-\frac{1}{2}\right) = \left[\frac{1}{2n}+\frac{1}{m}-\frac{1}{n}\right] - \frac{1}{2m} = \frac{1}{2m}-\frac{1}{2n} =$$

$$= \frac{1}{2}\left[\frac{1}{m}-\frac{1}{n}\right] \to 0$$

as $m, n \to +\infty$.

f_n is Cauchy $\|f_n - f_m\|_1 = \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right|$



Nevertheless f_n is NOT CONVERGING in $\|\cdot\|_1$ to a function f in X.

We claim $f_n \longrightarrow f = \begin{cases} 0 & x < \frac{1}{2} \\ 1 & x \geq \frac{1}{2} \end{cases}$ in $\|\cdot\|_1$

$$d(f_n, f) = \|f_n - f\|_1 = \int_0^1 |f_n(x) - f(x)|\,dx = \int_{\frac{1}{2}}^{\frac{1}{2}-\frac{1}{n}} 1 - n\left(x - \frac{1}{2}\right) =$$

$$= \frac{1}{2n} \longrightarrow 0 \quad \text{as } n \to +\infty$$

$f \notin X$ SINCE f is NOT CONTINUOUS!

$(X, \|\cdot\|_1)$ IS NOT COMPLETE

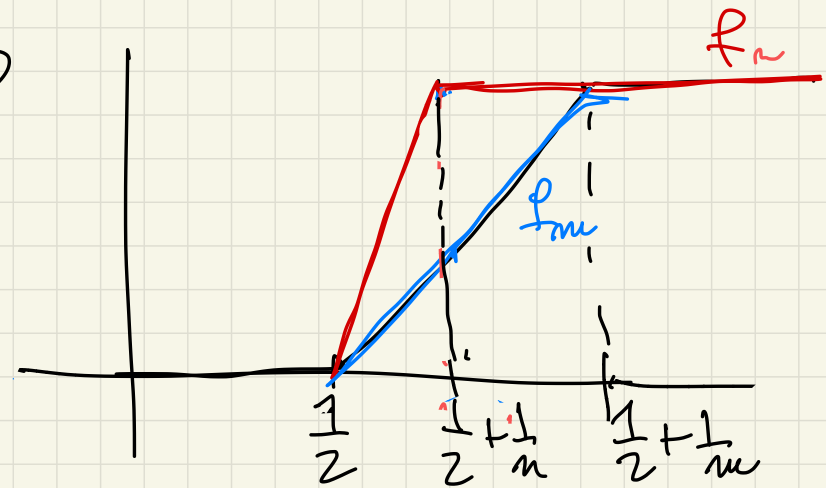
Observe that f_n is NOT CAUCHY WITH RESPECT TO $\|\cdot\|_\infty$!

$n > m$

$$\|f_n - f_m\|_\infty = \sup_{[0,1]} |f_n(x) - f_m(x)| \geq f_n\left(\frac{1}{2}+\frac{1}{n}\right) - f_m\left(\frac{1}{2}+\frac{1}{n}\right)$$

$$= 1 - \frac{m}{n} = \frac{n-m}{n} \not\to 0$$

$\boxed{m < n}$



$$\lim_{n,m\to+\infty} \|f_n - f_m\|_\infty =$$

$$= \lim_{n,m\to+\infty} \frac{n-m}{n} \not= 0$$

(take $m = \frac{n}{2}$ $\quad \|f_n - f_{\frac{n}{2}}\|_\infty = \frac{1}{2} \quad \forall n$

$m = \frac{n}{3} \quad \|f_n - f_{\frac{n}{3}}\|_\infty = \frac{2}{3} \quad \forall n$

One can prove that

$$\left(X = \{ f : [0,1] \to \mathbb{R} \text{ continuous} \}, \ \|\cdot\|_\infty \right)$$

is COMPLETE.

f_n Cauchy $\quad \|f_n - f_m\|_\infty = \sup_{x \in [0,1]} |f_n(x) - f_m(x)| \longrightarrow 0$

So $\forall x \in [0,1] \quad |f_n(x) - f_m(x)| \longrightarrow 0 \Rightarrow \exists \lim_n f_n(x) = f(x)$

One can also prove that f is continuous.