# COMPUTABILITY (19/11/2024)

OBSERVATION: A function which is total and not computable

$$f(x) = \begin{cases} \boxed{\varphi_x(x) + 1} & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \Psi_U(x,x) + 1 & \boxed{\text{if } \varphi_x(x)\downarrow} \\ 0 & \boxed{\text{otherwise}} \end{cases}$$

HALTING PROBLEM : The predicate below is UNDECIDABLE

$$Halt(x) = \begin{cases} true & \text{if } \varphi_x(x)\downarrow & (\text{i.e. } x \in W_x) \\ false & \text{if } \varphi_x(x)\uparrow & (\text{i.e. } x \notin W_x) \end{cases}$$

idea : by contradiction : we show that assuming Halt(x) decidable we can prove f computable

$$f(x) = \begin{cases} \Psi_U(x,x) + 1 & \text{if } \overbrace{\varphi_x(x)\downarrow}^{Halt(x)} \\ 0 & \underbrace{\text{otherwise}}_{\neg Halt(x)} \end{cases}$$

$$\cancel{=} \underbrace{(\Psi_U(x,x) + 1) \cdot \underbrace{\chi_{Halt}(x)}_{\begin{matrix} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{matrix}}}_{\begin{matrix} \text{if } \varphi_x(x)\downarrow & \rightsquigarrow & \varphi_x(x)+1 \\ \text{if } \varphi_x(x)\uparrow & \rightsquigarrow & \uparrow \end{matrix}}$$

instead

$$f(x) = \text{"}\Big(\mu(t,y,z).\ \big(S(x,x,y,t) \wedge (z = y+1) \wedge Halt(x)\big) \vee \big(z = 0 \wedge \neg Halt(x)\big)\Big)_z\text{"} \quad \begin{matrix} \rightsquigarrow y+1 \\ \rightsquigarrow 0 \end{matrix}$$

$$= \Big( \underset{\substack{t = (\omega)_1 \\ y = (\omega)_2 \\ z = (\omega)_3}}{\mu \omega} . \ \big( S(x, x, (\omega)_2, (\omega)_1) \wedge ((\omega)_3 = (\omega)_2 + 1) \wedge Halt(x) \big) \vee \big( (\omega)_3 = 0 \wedge \neg Halt(x) \big) \Big)_3$$

Observe that

$$Q(x, \omega) = \big( S(x, x, (\omega)_2, (\omega)_1) \wedge ((\omega)_3 = (\omega)_2 + 1) \wedge Halt(x) \big) \vee \big( (\omega)_3 = 0 \wedge \neg Halt(x) \big)$$

decidable

then

$$f(x) = \Big( \mu \omega . \ | \chi_{Q(x,\omega)} - 1 | \Big)_3$$

computable since it is the minimalisation of a computable function

$\Rightarrow$ contradiction

$\Rightarrow$ $Halt(x)$ is not decidable $\square$

**EXERCISE** (Definition by cases)

Let $Q(x)$ decidable predicate, $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ computable functions

and define

$$f(x) = \begin{cases} f_1(x) & \text{if } Q(x) \\ f_2(x) & \text{if } \neg Q(x) \end{cases}$$

Then $f$ is computable

proof

If $f_1, f_2$ are total then

$$f(x) = \underbrace{f_1(x) \cdot \underbrace{\chi_Q(x)}_{\substack{1 \text{ if } Q(x) \\ 0 \text{ otherwise}}}}_{\substack{f_1(x) \text{ if } Q(x) \\ 0 \text{ otherwise}}} + \underbrace{f_2(x) \cdot \underbrace{\chi_{\neg Q}(x)}_{\substack{1 \text{ if } \neg Q(x) \\ 0 \text{ otherwise}}}}_{\substack{f_2(x) \text{ if } \neg Q(x) \\ 0 \text{ otherwise}}}$$

$\Rightarrow$ $f$ computable as it is the composition of total function.

For general $f_1, f_2$, take $e_1, e_2 \in \mathbb{N}$ s.t. $f_1 = \varphi_{e_1}$ $f_2 = \varphi_{e_2}$

$$f(x) = \text{"}\left(\mu(y,t).\left( (S(e_1,x,y,t) \wedge Q(x)) \vee (S(e_2,x,y,t) \wedge \neg Q(x)) \right)\right)_y\text{"}$$

$$= \left(\underset{\substack{(\omega)_1 = y \\ (\omega)_2 = t}}{\mu \omega}. \underbrace{\left( (S(e_1, x, (\omega)_1, (\omega)_2) \wedge Q(x)) \vee (S(e_2, x, (\omega)_1, (\omega)_2) \wedge \neg Q(x)) \right)}_{\text{decidable}}\right)_1$$

$\Rightarrow$ $f$ computable

**EXERCISE** : TOTALITY

$Tot(x) \equiv$ "$\varphi_x$ is total" $\equiv$ "$P_x$ halts on every input"

is undecidable.

In fact

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } Tot(x) \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ $f$ is total $\left( \begin{array}{lll} \text{if } Tot(x) & \rightsquigarrow & f(x) = \overset{\downarrow}{\varphi_x(x)} + 1 \downarrow \\ \text{if } \neg Tot(x) & \rightsquigarrow & f(x) = 0 \end{array} \right)$

$\rightarrow$ $f$ not computable (different from all total computable functions

in fact $\forall x$ if $\varphi_x$ is total then $f(x) = \varphi_x(x) + 1 \neq \varphi_x(x)$ )

$\rightarrow$ if $Tot(x)$ were decidable then $f$ would be computable

$\Rightarrow$ $Tot$ not decidable

In fact

$$f(x) = \begin{cases} f_1(x) & \text{if } Tot(x) \\ f_2(x) & \text{if } \neg Tot(x) \end{cases}$$

where $f_1(x) = \varphi_x(x) + 1 = \Psi_U(x,x) + 1$ computable

$f_2(x) = 0 \quad \forall x$ "

hence by previous exercise $f$ computable, absurd.

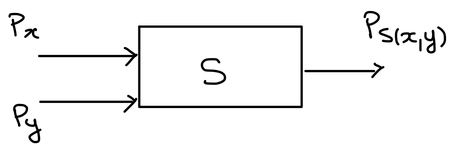
$\square$

## EFFECTIVE OPERATIONS ON COMPUTABLE FUNCTIONS

① there a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$

$\forall x, y$

$$\varphi_{s(x,y)}(z) = \varphi_x(z) * \varphi_y(z) \qquad \forall z$$




def $P_{s(x,y)}(z)$:

$v_x = P_x(z)$

$v_y = P_y(z)$

return $v_x * v_y$

define $g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x, y, z) = \varphi_x(z) * \varphi_y(z) = \Psi_U(x, z) * \Psi_U(y, z)$$

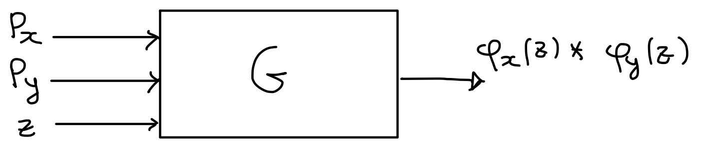
computable (composition of computable functions)

by smn theorem (corollary of) there is $s : \mathbb{N}^2 \to \mathbb{N}$ total computable

such that

$$\varphi_{s(x,y)}(z) = g(x, y, z) = \varphi_x(z) * \varphi_y(z)$$

as desired.




function $s$ inputs programs $P_x$ and $P_y$ and produces a program obtained by G "hard coding" $P_x$ and $P_y$

EXERCISE : Effectiveness of inverting a computable function

There is a total computable function $k : \mathbb{N} \to \mathbb{N}$ s.t.

$\forall x \in \mathbb{N}$ if $\varphi_x$ injective then $\varphi_{k(x)} = (\varphi_x)^{-1}$

$$P_x \longrightarrow \boxed{\;K\;} \longrightarrow P_{k(x)}$$

program for $(\varphi_x)^{-1}$

$(\varphi_x)^{-1}(y)$

$z=0$ $\varphi_x(0)=y$?

$z=1$ $\varphi_x(1)=y$?

$\vdots$

define

$$g(x,y) = (\varphi_x)^{-1}(y) = \begin{cases} z & \text{s.t. } \varphi_x(z)=y \text{ if it exists} \\ \uparrow & \text{otherwise} \end{cases}$$

$(\varphi_x)^{-1}$ injective

$$= \text{"}\left(\mu(z,t).\ S(x,z,y,t)\right)_z\text{"}$$

$$= \left(\mu\omega.\ S(x, (\omega)_1, y, (\omega)_2)\right)_1$$

$$= \left(\mu\omega.\ |\chi_S(x, (\omega)_1, y, (\omega)_2) - 1|\right)_1$$

$\Rightarrow$ $g$ is computable

Hence by smn theorem there is $k : \mathbb{N} \to \mathbb{N}$ total computable s.t.

$\forall x$

$$\varphi_{k(x)}(y) = g(x,y) = (\varphi_x)^{-1}(y) \quad \text{if } \varphi_x \text{ is injective}$$

What if $\varphi_x$ is <u>not</u> injective ?

↳ $\varphi_{k(x)}(y)$ is one of the possible counter images of $y$ (if any)

QUESTION : Given $f : \mathbb{N} \to \mathbb{N}$ computable, let $f^{-1}(y) = \{x \mid f(x) = y\}$

Define

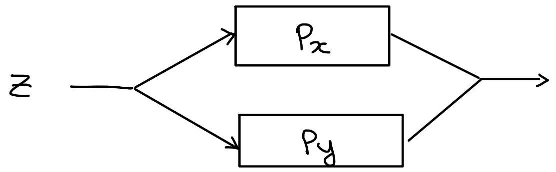
$$g(y) = \begin{cases} \min f^{-1}(y) & \text{if } f^{-1}(y) \neq \emptyset \\ \uparrow & \text{otherwise} \end{cases}$$

Is $g$ computable ?

**EXERCISE**. There is a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$

such that $$W_{s(x,y)} = W_x \cup W_y$$

$$\forall z. \quad \varphi_{s(x,y)}(z)\downarrow \quad \text{iff} \quad \varphi_x(z)\downarrow \quad \text{or} \quad \varphi_y(z)\downarrow$$




$$g : \mathbb{N}^3 \to \mathbb{N}$$

$$g(x,y,z) = \begin{cases} \cancel{\downarrow}\ 1 & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mathbb{1}\left(\mu t.\ H(x,z,t) \vee H(y,z,t)\right)$$

$$\mathbb{1} : \mathbb{N} \to \mathbb{N} \qquad \mathbb{1}(x) = 1 \quad \forall x$$

function $g$ is computable, hence by smn theorem $\exists\, s : \mathbb{N}^2 \to \mathbb{N}$ total computable s.t. $\forall x, y, z$

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \begin{cases} 1 & \text{if } \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Hence $s$ is the desired function

$$z \in W_{s(x,y)} \quad \text{iff} \quad \underset{\underset{g(x,y,z)}{\parallel}}{\varphi_{s(x,y)}(z)}\downarrow \quad \text{iff} \quad \varphi_x(z)\downarrow \text{ or } \varphi_y(z)\downarrow$$

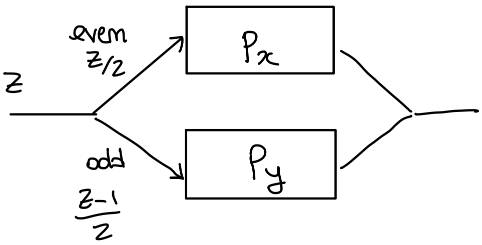
$$\text{iff} \quad z \in W_x \text{ or } z \in W_y$$

$$\text{iff} \quad z \in W_x \cup W_y$$

EXERCISE : There is a total computable function $s: \mathbb{N}^2 \to \mathbb{N}$ such that

$$E_{s(x,y)} = E_x \cup E_y$$

( $P_{s(x,y)}$ produces in output all outputs of $P_x$ and $P_y$ and only )




| | 0 | 1 | 2 | 3 | --- |
|---|---|---|---|---|---|
| $P_x$ | 4 | 1 | ↑ | 5 | -- |
| $P_y$ | 2 | ↑ | 7 | 2 | ... |

$$g(x,y,z) = \begin{cases} \varphi_x(z/2) & \text{if } z \text{ even} \\ \varphi_y\left(\frac{z-1}{2}\right) & \text{if } z \text{ odd} \end{cases}$$

$$\cancel{= \Psi_U(x, qt(2,z)) * \overline{sg}(rm(2,z)) + \Psi_U(y, qt(2,z)) * rm(2,z)}$$

$$= \Big(\mu(v,t). \ \big( (S(x,z,v,t) \wedge z \text{ even}) \vee (S(y,z,v,t) \wedge z \text{ odd}) \big)\Big)_v$$

$$= \Big(\mu w. \ \underbrace{(S(x, z, (w)_1, (w)_2) \wedge z \text{ even}) \vee S(y, z, (w)_1, (w)_2) \wedge z \text{ odd})}_{\text{decidable}}\Big)_1$$

computable.

By smn theorem there is $s: \mathbb{N}^2 \to \mathbb{N}$ total computable s.t.

$$\varphi_{s(x,y)}(z) = g(x,y,z) = \begin{cases} \varphi_x(z/2) & \text{if } z \text{ even} \\ \varphi_y\left(\frac{z-1}{2}\right) & \text{if } z \text{ odd} \end{cases}$$

I claim : $s$ is the desired function

$$\boxed{E_{s(x,y)} = E_x \cup E_y}$$

$(\subseteq)$ $\quad v \in E_{s(x,y)}$

$\exists z$ s.t. $\quad \varphi_{s(x,y)}(z) = v$

$$\varphi_{s(x,y)}(z) = g(x,y,z)$$

we have two possibilities

$$\left.\begin{array}{l} - \ v = \varphi_x(z/2) \Rightarrow v \in E_x \\ - \ v = \varphi_y\left(\frac{z-1}{2}\right) \Rightarrow v \in E_y \end{array}\right\} \rightarrow v \in E_x \cup E_y$$

$(\supseteq)$ $\quad v \in E_x \cup E_y$ then $v \in E_{s(x,y)}$

i.e. (1) $v \in E_x \rightarrow v \in E_{s(x,y)}$

(2) $v \in E_y \rightarrow v \in E_{s(x,y)}$

e.g.

(1) $v \in E_x$ i.e. there is $z \in \mathbb{N}$ s.t. $v = \varphi_x(z)$

therefore $\varphi_{s(x,y)}(2z) = \varphi_x(\cancel{2}z/\cancel{2}) = \varphi_x(z) = v$

(2) same.

<u>EXERCISE</u> : variant of URM

URM$^P$

$Z(n)$

~~$S(n)$~~ $\quad P(n) \qquad r_n \leftarrow r_n \dot{-} 1$

$T(m,n)$

$J(m,n,t)$

$\mathcal{C}^P$ URM$^P$ computable functions vs $\mathcal{C}$

<u>EXERCISE</u> : Are there $f, g : \mathbb{N} \rightarrow \mathbb{N}$ functions s.t.

(1) $f$ computable, $g$ not computable $\quad f \circ g$ computable ?

(2) $f$ not computable, $g$ not computable $\quad f \circ g$ computable ?