Continuous and compact embeddings in Sobolev.

CONTINUOUS EMBEDDINGS

$p < n$, $p = n$, $p > n$, $p = +\infty$

GAGLIARDO NIRENBERG SOBOLEV INEQUALITY.

Let $p \in [1, n)$ $\quad p^* =$ Sobolev conjugate $\Rightarrow$ $\boxed{\frac{1}{p} - \frac{1}{n} = \frac{1}{p^*}}$

$$p^* = \frac{pn}{n-p} > p.$$

$C(n,p) = \frac{n-1}{n-p} \cdot p$

$\exists\ C = C(p,n)$ (explicit!) such that

$$\| f \|_{L^{p^*}(\mathbb{R}^n)} \leq C(p,n) \underbrace{\| |Df| \|_{L^p(\mathbb{R}^n)}}$$

$$= \Big\| \sum_{i=1}^{n} \Big| \frac{\partial f}{\partial x_i} \Big| \Big\|_{L^p} \underset{\text{MINKOWSKI}}{\leq} \sum_{i=1}^{n} \Big\| \frac{\partial f}{\partial x_i} \Big\|_{L^p}$$

IN PARTICULAR $\quad \boxed{W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n) \quad \text{CONTINUOUS EMB.}}$

Corollary 1 $\forall\, q \in [p, p^*]$ $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ continuous embedding

$\exists\, C = C(n,p)$ such that $\|f\|_{L^q(\mathbb{R}^n)} \le C \|f\|_{W^{1,p}(\mathbb{R}^n)}$

proof $f \in W^{1,p}(\mathbb{R}^n) \to f \in L^p(\mathbb{R}^n)$
GNS $\Rightarrow f \in L^{p^*}(\mathbb{R}^n)$

INTERPOLATION $f \in L^q(\mathbb{R}^n)$ $\forall q \in [p, p^*]$

$$\|u\|_{L^q(\mathbb{R}^n)} \underset{\text{generalized Holder}}{\le} \|u\|^{\theta}_{L^p(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^{p^*}(\mathbb{R}^n)} \underset{\text{GNS}}{\le} \underbrace{\|u\|^{\theta}_{L^p} [C(n,p)]^{1-\theta} \| |Du| \|^{1-\theta}_{L^p}}_{\text{YOUNG}}$$

$$\frac{\theta}{p} + \frac{(1-\theta)}{p^*} = \frac{1}{q}$$

$$\le \|u\|_{L^p} \cdot \theta + (1-\theta) C(n,p) \| |Du| \|_{L^p}$$

$$\le [1 + C(n,p)] \|u\|_{W^{1,p}}$$

GENERALIZED HOLDER $\quad \frac{\theta}{p} + \frac{1-\theta}{p^*} = \frac{1}{q} \quad q \in (p, p^*)$

CONJUGATE of $\frac{p}{\theta q}$ is $\frac{p^*}{(1-\theta) q}$

$$|u|^q = \underset{\in L^{\frac{p}{\theta q}}}{|u|^{\theta q}} \ \underset{\in L^{\frac{p^*}{(1-\theta) q}}}{|u|^{(1-\theta) q}}$$

APPLY HOLDER

$$\int_{\mathbb{R}^n} |u|^q = \int_{\mathbb{R}^n} |u|^{\theta q} |u|^{(1-\theta) q} \le \|u\|^{\theta q}_{L^p} \|u\|^{(1-\theta) q}_{L^{p^*}} \Longrightarrow \|u\|_{L^q} \le \|u\|^{\theta}_{L^p} \|u\|^{1-\theta}_{L^{p^*}}$$

why p^*? guess it is the correct exp. by rescaling,
if (GNS) is true for u, then is true for $u_\lambda(x) = u(\lambda x)$

$$\|u\|_{L^q} \leq C \|Du\|_{L^p}$$

$$\Downarrow$$

$$\|u_\lambda\|_q = \left[\int_{\mathbb{R}^n} |u(\lambda x)|^q dx\right]^{1/q} = \lambda^{-\frac{n}{q}} \|u\|_q$$

$$\|Du_\lambda\|_p = \left[\int_{\mathbb{R}^n} \lambda^p |Du(\lambda x)|^p dx\right]^{1/p} = \lambda^{1-\frac{n}{p}} \|Du\|_p$$

$$1-\frac{n}{p} = -\frac{n}{q} \Rightarrow \frac{1}{q} = \frac{1}{p} - \frac{1}{n}$$

Lemma Let $v_1 \dots v_n \in C_c^\infty(\mathbb{R}^{n})$, $v_i \geq 0$, $n \geq 1$, v_i DOES NOT DEPEND on x_i

$$\left\| \prod_{i=1}^n v_i \right\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^n \|v_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$$

proof by induction on the dimension n

$n=2$
$$\int_{\mathbb{R}^2} v_1(x_2)\, v_2(x_1)\, dx_1 dx_2 = \int_{\mathbb{R}} v_1(x_2) \int_{\mathbb{R}} v_2(x_1)$$
Fubini Tonelli

NO PROOF DURING CLASS

If one is interested I put here the proof. Assume true for dimension n.

$u_1 \dots u_{n+1}$, u_{n+1} NOT DEPENDING ON x_{n+1}. $x' = x_1 \dots x_{n-1}$

$$\|u_1 \dots u_{n+1}\|_{L^1} = \int_{\mathbb{R}^{n+1}} \prod_{i=1}^{n+1} u_i = \int_{\mathbb{R}^n} u_{n+1} \int_{\mathbb{R}} \prod_{i=1}^{n} u_i \, dx_n \, dx' \le \text{HÖLDER } p=n \ p'=\frac{n}{n-1}$$

$$\le \|u_{n+1}\|_{L^n(\mathbb{R}^n)} \left[\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}} \prod_{i=1}^{n} u_i \, dx_n \right]^{\frac{n}{n-1}} dx' \right]^{\frac{n-1}{n}} \quad (*)$$

Recall generalized Hölder: $v_1 \in L^{p_1} \dots v_k \in L^{p_k}$ $\quad \frac{1}{p_1} + \dots + \frac{1}{p_k} = 1$

$$\int \prod_i |v_i| \le \|v_1\|_{p_1} \cdot \ldots \cdot \|v_k\|_{p_k}.$$

$u_i \in L^n(\mathbb{R})$

$$\int_{\mathbb{R}} \prod_i u_i \, dx_n \le \prod_{i=1}^{n} \left[\int_{\mathbb{R}} u_i^n \, dx_n \right]^{\frac{1}{n}}$$

$$(*) \le \|u_{n+1}\|_{L^n(\mathbb{R}^n)} \left[\int_{\mathbb{R}^n} \prod_{i=1}^{n} \left[\int_{\mathbb{R}} u_i^n \, dx_n \right]^{\frac{1}{n-1}} dx' \right]^{\frac{n-1}{n}}$$

INDUCTIVE ASSUMPTION on $v_i = \left[\int_{\mathbb{R}} u_i^n \, dx_n \right]^{\frac{1}{n-1}}$

v_i does not depend on x_i (and on x_n)

$$\le \|u_{n+1}\|_{L^n} \prod_{i=1}^{n} \|v_i\|_{L^{n-1}(\mathbb{R}^{n-1})} = \prod_{i=1}^{n} \|u_i\|_{L^n}$$

$$\|v_i\|_{L^{n-1}} = \left[\int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} u_i^n \right] dx' \right]^{\frac{1}{n}} = \int_{\mathbb{R}^n} u_i^n = \|u_i\|_{L^n}.$$

proof of GNS Obs $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$ $(p<n!)$

If we prove (GNS) $\forall \phi \in C_c^\infty(\mathbb{R}^n)$ $\to$ Let $f \in W^{1,p}(\mathbb{R}^n)$ and

$\phi_k \to f$ in $W^{1,p}$ $\phi_k \in C_c^\infty(\mathbb{R}^n) \Rightarrow \phi_k$ is Cauchy in $W^{1,p}$

$$\|\phi_k - \phi_l\|_{L^{p^*}} \le C(n,p)\,\|D(\phi_k - \phi_l)\|_{L^p} \longrightarrow 0$$

$\Rightarrow \phi_k$ is Cauchy in L^{p^*} $\Rightarrow$ $\phi_k \to f$ in L^{p^*} $\Rightarrow \|\phi_k\|_{L^{p^*}} \le C(n,p)\|D\phi_k\|_{L^p}$

$\|f\|_{L^{p^*}} \le C(n,p)\,\|Df\|$

So prove just for $C_c^\infty(\mathbb{R}^n)$

$p=1$ $\quad 1^* = 1 - \frac{1}{n} = \frac{n-1}{n}$ $\quad f \in C_c^\infty(\mathbb{R}^n)$

$$|f(\bar{x}_1 \dots \bar{x}_n)| = \left|\int_{-\infty}^{\bar{x}_i} \frac{\partial}{\partial x_i} f(\bar{x}_1, \dots t, \dots \bar{x}_n)\, dt\right| \le \int_{-\infty}^{+\infty} \left|\frac{\partial f}{\partial x_i}(\dots t \dots)\right| dt$$

$$|f(x)|^n \le \prod_{i=1}^n \int_{\mathbb{R}} \left|\frac{\partial f}{\partial x_i}(\dots \overset{i}{t} \dots)\, dt\right|$$

$$|f(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^n \left[\int_{\mathbb{R}} \left|\frac{\partial f}{\partial x_i}(\dots t)\, dt\right|\right]^{\frac{1}{n-1}}$$

$\to$ INTEGRATE IN $\mathbb{R}^n$ and use LEMMA with

$$v_i(x) = \left[\int_{\mathbb{R}} \left|\frac{\partial f}{\partial x_i}(\dots t \dots)\, dt\right|\right]^{\frac{1}{n-1}}$$

$$\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dx = \|f\|_{L^{n/n-1}}^{n-1/n}$$

$$v_i = \left[\int_{\mathbb{R}} \left|\frac{\partial f}{\partial x_i}(\ldots t \ldots)\right| dt\right]^{\frac{1}{n-1}} \qquad \int_{\mathbb{R}^n} \Pi_i v_i \le \prod_{i=1}^n \|v_i\|_{L^{n-1}} = \prod_{i=1}^n \left[\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left|\frac{\partial f}{\partial x_i}\right|\right]^{\frac{1}{n-1}} =$$

$$= \prod_{i=1}^n \left\|\frac{\partial f}{\partial x_i}\right\|_{L^1(\mathbb{R}^n)}^{\frac{1}{n-1}} \le \| |Df| \|_{L^1(\mathbb{R}^{n-1})}^{\frac{n}{n-1}}$$

$$\|f\|_{L^{n/n-1}}^{\cancel{\frac{n-1}{n}}} \le \| |Df| \|_{L^1}^{\cancel{\frac{n}{n-1}}} \qquad \square \qquad \text{Obs. } C(n,1) = 1$$

$p > 1$ take $\gamma > 0$ to be fixed later $|f(x)|^\gamma \in C_c^\infty(\mathbb{R}^n)$

$$|D|f(x)|^\gamma| = \gamma\, |f(x)|^{\gamma-1} |Df(x)|$$

Apply (GNS) to $|f(x)|^\gamma$ for $p = 1$

$$\|f\|_{L^{\frac{n}{n-1}\gamma}}^{\gamma} = \||f|^\gamma\|_{L^{\frac{n}{n-1}}} \overset{\text{GNS}}{\le} \gamma \int_{\mathbb{R}^n} \underbrace{|f(x)|^{\gamma-1}}_{\in L^{p'}} \underbrace{|Df(x)|}_{\in L^p} \overset{\text{Holder}}{\le} \gamma \||Df|\|_{L^p} \left[\int_{\mathbb{R}^n} |f(x)|^{(\gamma-1)\cdot\frac{p}{p-1}} dx\right]^{\frac{p-1}{p}}$$

$$\|f\|_{L^{n\gamma/n-1}}^{\gamma} \le \gamma \||Df|\|_{L^p} \|f\|_{L^{(\gamma-1)(p-1)/p}}^{\gamma-1}$$

choose γ such that $(\gamma-1)\frac{p}{p-1} = \frac{n\gamma}{n-1} \Rightarrow \boxed{\gamma = \left(\frac{n-1}{n-p}\right)\cdot p}$

for such γ $\quad (\gamma-1)\frac{p}{p-1} = \frac{n\gamma}{n-1} = \frac{n}{(n-p)}p = \frac{np}{n-p} = p^*$

$$\|f\|_{L^{p^*}} \le \left(\frac{n-1}{n-p}\right)p \; \| |Df| \|_{L^p} \qquad C(n,p) = \frac{(n-1)}{(n-p)}p \left(\begin{matrix}\to \infty \\ p\to n\end{matrix}\right)$$

Remark When $p = n$? $\quad$ ⊛ $\|f\|^{\gamma}_{L^{\frac{n}{n-1}\gamma}} \le \gamma \cdot \|Df\|_{L^n} \cdot \|f\|^{\gamma-1}_{L^{(\gamma-1)\frac{n}{n-1}}}$

choose $\gamma = n$ in ⊛ $\quad \|f\|^{n}_{n^2/n-1} \le n \; \| |Df| \|_{L^n} \cdot \|f\|^{n-1}_{L^n}$.

$$\boxed{\|f\|_{n^2/n-1} \le n^{\frac{1}{n}} \; \|f\|_{W^{1,n}}}$$

$W^{1,n} \hookrightarrow L^p \quad \forall p \in [n, \frac{n^2}{n-1}]$
continuously

choose $\gamma = n+1$ in ⊛ $\quad \|f\|^{n+1}_{L^{\frac{n(n+1)}{n-1}}} \le (n+1) \; \| |Df| \|_{L^n} \; \|f\|^{n}_{L^{n^2/n-1}}$

by previous embedd.

$$\|f\|^{n+1}_{L^{\frac{n(n+1)}{n-1}}} \le (n+1) \; \| \, |Df| \, \|_{L^n} \; n \; \|f\|^n_{W^{1,n}}$$

$$\|f\|_{L^{\frac{n(n+1)}{n-1}}} \le \left[(n+1)\cdot n\right]^{\frac{1}{n+1}} \|f\|_{W^{1,n}}$$

$$W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \quad \text{continuously} \quad \forall q \in \left[n, \frac{n(n+1)}{n-1}\right]$$

choose $\gamma = n+2$

$\longrightarrow$ $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) \quad \forall q \in [n, +\infty)$ CONTINUOUSLY.

But $W^{1,n}(\mathbb{R}^n) \not\hookrightarrow L^\infty(\mathbb{R}^n)$

Ex $f(x) = \lg\left(\lg\left(1+\frac{1}{|x|}\right)\right) \in W^{1,n}(B(0,1)) \rightsquigarrow$ Extend $\bar{f} \in W^{1,n}(\mathbb{R}^n)$ $(\operatorname{supp} \bar{f} \subseteq B(0,2))$

$f \notin L^\infty(\mathbb{R}^n)$.

Local version of GNS for $W_0^{1,p}$

(1) U bdd open in $\mathbb{R}^n$ $\quad p<n$

then $\forall f \in W_0^{1,p}(U)$, $\forall q \in [1,p^*]$ $\exists\, C = C(n,p,q,U)$ such that

$$\|f\|_{L^q} \le C \,\|\,|Df|\,\|_{L^p}$$

Obs in particular $\|f\|_{L^p} \le C(n,p)\,\|\,|Df|\,\|_{L^p}$ $\rightarrow$ SOMETIMES CALLED POINCARÉ INEQ.

$\Rightarrow$ $\|\,|Df|\,\|_{L^p}$ is an EQUIVALENT NORM to $\|\cdot\|_{1,p}$ in $W_0^{1,p}(U)$ (U bdd).

proof Sufficient to show for $f \in C_c^\infty(U) \subseteq C_c^\infty(\mathbb{R}^n)$.

by (GNS) $\forall f \in C_c^\infty(U)$ $\quad \underbrace{\|f\|_{L^{p^*}(U)} \le C(n,p)\,\|\,|Df|\,\|_{L^p}}$

$\forall f \in W_0^{1,p}(U)$.

U bdd so $f \in L^{p^*}(U) \Rightarrow f \in L^q(U)$ $\forall q \in [1,p^*]$

$$\|f\|_q^q = \int_U |f|^q\,dx \le \|f\|_{L^{p^*}}^q\,|U|^{1-\frac{q}{p^*}} \Longrightarrow \|f\|_{L^q} \le |U|^{\frac{p^*-q}{p^* q}} \cdot \|f\|_{L^{p^*}} \le C\,\|\,|Df|\,\|_{L^p}$$

$q<p^*$ $\quad$ HÖLDER $\frac{p^*}{q}, \left(\frac{p^*}{q}\right)'$