Extension theorem

$U \subseteq \mathbb{R}^n$ open bdd of class C^1.

Let $V \subseteq \mathbb{R}^n$ bdd $U \subset\subset V$. $p \in [1, +\infty]$

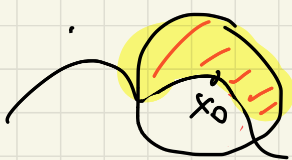
$\exists \; E : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$ such that

1) $E(f) = f$ a.e in U
2) $\|E(f)\|_{W^{1,p}(\mathbb{R}^n)} \leq C \, \|f\|_{W^{1,p}(U)}$ $\qquad C = C(U, V, n, p)$
3) $\operatorname{supp} E(f) \subseteq V \quad \forall f \in W^{1,p}(U)$.

Proof of extension theorem

Hp U is bdd open set of class C^1. Fix V open bdd, $U \subset\subset V$.

$\forall x_0 \in \partial U \; \exists r_0$ such that $B(x_0, r_0) \cap U = \{ x_n > \gamma(x_1 \dots x_{n-1}) \}$

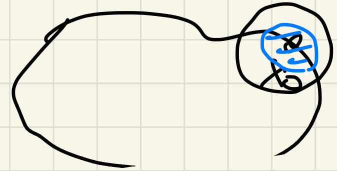


$\xrightarrow{\Phi \; C^1 \text{ diffeom}}$

$\Phi(B(x_0, r_0) \cap U) =$

$= \{ y_n < 0 \} \cap U'(y_0)$

we consider in every $x_0 \in \partial U$ the C^1 diffeomorphism which flattens locally the bdry $\rightarrow$ if $u \in W^{1,p}(U)$ then $u \in W^{1,p}(B(x_0,r_0)\cap U)$, we consider $\bar{u}(y) = u(\Phi(x))$ and extend it by the lemma $\Rightarrow$ come back to the initial coordinates and we have extension of u in $B(x_0, r_1)$ for some $r_1 < r_0$



$\Rightarrow \bar{u}_{x_0} \in W^{1,p}(B(x_0,r_1)\cap U)$

$\|\bar{u}_{x_0}\|_{W^{1,p}} \leq C\|u\|_{W^{1,p}}$

$\bar{u}_{x_0} \in W^{1,p}(B(x_0,r_1)\cap U)$

$\|\bar{u}_{x_0}\|_{W^{1,p}} \leq C\|u\|_{W^{1,p}(U)}$ where C depends on Φ (C^1 diffeo at x_0)

$\forall x_0 \in \partial U \rightarrow$ consider $r_1^{x_0} \longrightarrow \bar{r}_1^{x_0} \leq \min\left(\frac{r_1^{x_0}}{2}, \frac{\text{dist}(U, \mathbb{R}^n \setminus V)}{2}\right)$

cover ∂U with $B(x_0, \bar{r}_1^{x_0}) \cap \partial U \Rightarrow$ extract a finite cover

$B(x_0^i, \bar{r}_1^i) \Rightarrow U = \bigcup_{i=1}^{M} [B(x_0^i, \bar{r}_1^i)\cap U] \cup U_0$ where $U_0 = U \setminus \overline{\bigcup_{i=1}^{M} (B(x_0^i, \frac{\bar{r}_1^i}{2})}$

take $(\xi_i) \in C_c^\infty(\mathbb{R}^n)$ partition of units $0 \leq \xi_i \leq 1$

$\text{supp}\ \xi_0 \subseteq U_0$ $\quad$ $\text{supp}\ \xi_i \subseteq B(x_0^i, \bar{r}_1^i)$, $\quad \sum_{i=0}^{M} \xi_i(x) = 1 \quad \forall x \in U$

$u = (\sum_i \xi_i) \cdot u$

$\bar{u} := u\xi_0 + \sum_{i=1}^{M} \bar{u}_i \xi_i$

$\bar{u}_i$ is the extension on $B(x_0^i, r_1^i)$ $\quad \bar{u}_i \in W^{1,p}(B(x_0^i, r_1^i))$

$\bar{u}_i \cdot \xi_i \equiv 0$ outside $B(x_0^i, \bar{r}_1^i)$

$\bar{u}_i \in W^{1,p}(\mathbb{R}^n)$ $\quad \bar{r}_1^i < \frac{r_1^i}{2}$

$\boxed{\bar{u} \in W^{1,p}(\mathbb{R}^n)}$ $\quad$ $\boxed{Eu = \bar{u}}$

$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq \bar{C} \|u\|_{W^{1,p}(U)}$ where C depends on U

$\text{supp}\ \bar{u} \subseteq U_0 \cup \bigcup_{i=1}^{M} B(x_0^i, \bar{r}_1^i) \subset V$ since $\bar{r}_1^i \leq \frac{\text{dist}(x, \mathbb{R}^n \setminus V)}{2}$

NB the extension holds for $p \in [1, +\infty]$

Simple obs

Same argument as the extension lemma works for

$$U = \mathbb{R}^n_+ = \{(x_1 \dots x_n) \quad x_n > 0\} \qquad (x_1 \dots x_n) = (x', x_n)$$

$$E : W^{1,p}(\mathbb{R}^n_+) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

$$f \longrightarrow f^*(x', x_n) = \begin{cases} f(x', x_n) & x_n > 0 \\ f(x', -x_n) & x_n < 0 \end{cases}$$

$$\| f^* \|_{W^{1,p}(\mathbb{R}^n)} \doteq 2 \| f \|_{W^{1,p}(\mathbb{R}^n_+)}$$

EXTENSIONS for $W^{k,p}(U)$

$\boxed{k=2}$: If U bdd of class C^2, $V \subseteq \mathbb{R}^n$ $U \subset\subset V$ $\Longrightarrow$

$$E : W^{2,p}(U) \longrightarrow W^{2,p}(\mathbb{R}^n)$$

$$\Longrightarrow \begin{cases} Eu = u \quad \text{a.e in } U \\ \|Eu\|_{W^{2,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{2,p}(U)} \\ \text{supp } Eu \subseteq V \quad \forall u \in W^{1,p}(U). \end{cases}$$

+ USE LEMMA with more sophisticated reflection.

(NB U has to be of class C^2 $\Longleftrightarrow$ Φ diffeo flattening the boundary is of class C^2..)

$k>2$

If $k>2$ $W^{k,p}(U) \to W^{k,p}(\mathbb{R}^n)$ U bdd set of class C^k requires arguments more sophisticated.

TRACES

$p \in [1, +\infty)$ U bdd of class C^1 $\longrightarrow$ ∂U is a C^1 $(n-1)$ dim. manifold

$\exists\, T : W^{1,p}(U) \longrightarrow L^p(\partial U) = \{ f : \partial U \to \mathbb{R} \quad \int_{\partial U} |f|^p \, d\mathcal{H}^{n-1} < +\infty \}$

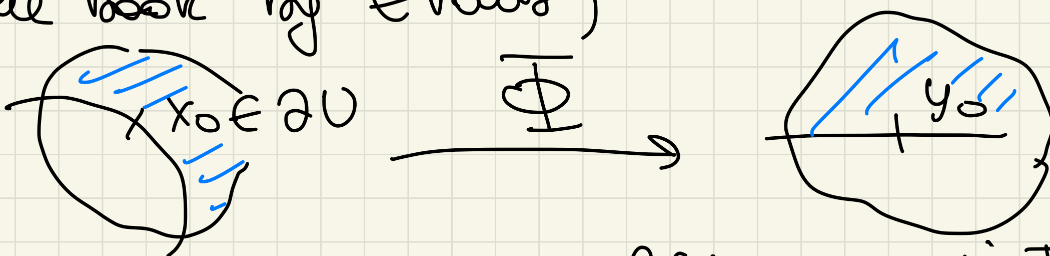
1) $Tf = f|_{\partial U}$ if $f \in W^{1,p}(U) \cap C(\overline{U})$

2) $\| Tf \|_{L^p(\partial U)} \leq C(p, U) \, \| f \|_{W^{1,p}(U)}$

same result for ∂U Lipschitz $(n-1)$ dim manifold ...

<u>no proof</u> (in the book by Evans)

idea is localize + flatten



$\partial \Phi(B(x_0, r_0) \cap U) = \{ y_n = 0 \}$

$f(y_1 - y_{n-1}, y_n) \xrightarrow{T} \tilde{f}(y_1 \dots y_{n-1}, 0)$

$Tf \in L^p(\mathbb{R}^{n-1} \cap \Phi(B(x_0, r_0)))$

Not all functions in $L^p(\partial U)$ are traces of functions in $W^{1,p}(U)$!

example $T(W^{1,2}(U)) = H^{1/2}(\partial U) = \{ f \in L^2(\partial U)$ such that $\underbrace{(1+|x|)\,\mathcal{F}(f)} \in L^2(\partial U)$

$\mathcal{F}$ = Fourier transf.

- Integration by parts in $W^{1,p}(U)$. / divergence thm.

$\forall\ \phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ $\quad \underbrace{(\text{NOT } \mathcal{C}^\infty_c(U))}$

if $\phi \in \mathcal{C}^\infty_c(U)$ this term DISAPPEARS

$$\int_U u \frac{\partial \phi}{\partial x_i}\, dx = -\int_U \frac{\partial u}{\partial x_i} \phi\, dx + \overbrace{\int_{\partial U} \phi\, T(u) \cdot \underbrace{\nu_i}_{\substack{i\text{-component} \\ \text{ext normal}}}\, dS}$$

Characterization of $W_0^{1,p}(U)$

U bdd of class C^1 $\qquad f \in W_0^{1,p}(U) \Longleftrightarrow Tf = 0$

$\Rightarrow$ easy $f_n \in C_c^\infty(U)$ $\quad Tf_n = 0 \quad \|f_n - f\|_{W^{1,p}} \to 0$

$$|T(f)| = |T(f_n - f)| \le C \cdot \|f_n - f\|_{W^{1,p}} \to 0.$$

$\Leftarrow$ technical see Evans.

RECALL

Let $p \in [1, +\infty)$ $\quad p' = \frac{p}{p-1} \in (1, +\infty]$

$$f_n \rightharpoonup f \text{ in } L^p \Longleftrightarrow \int_U f_n g \to \int_U f g \quad \forall g \in L^{p'}(U)$$

$p = +\infty$

$$f_n \overset{*}{\rightharpoonup} f \text{ in } L^\infty \Longleftrightarrow \int_U f_n g \to \int_U f g \quad \forall g \in L^1(U)$$

Assume $f_n \in W_0^{1,p}(U)$ $\quad f \in W^{1,p}(U)$

such that $f_n \longrightarrow f$ in $\underset{p<+\infty}{L^p}$ $\left(\text{or } f_n \overset{*}{\longrightarrow} f \text{ in } L^\infty\right)$

$$\frac{\partial}{\partial x_i} f_n \longrightarrow \frac{\partial}{\partial x_i} f \text{ in } \underset{p<+\infty}{L^p} \quad \left(\text{or } \frac{\partial f_n}{\partial x_i} \overset{*}{\longrightarrow} \frac{\partial f}{\partial x_i} \text{ in } L^\infty\right)$$

then $f \in W_0^{1,p}(U)$ $\quad$ [$W_0^{1,p}(U)$ is weakly closed in $W^{1,p}(U)$]

<u>proof</u> integration by part formula. Take $\phi \in C^\infty(\mathbb{R}^n)$

$\phi|_U \in L^q(U) \quad \forall q \in [1, +\infty]$

$$\int_U \frac{\partial \phi}{\partial x_i} f_n = -\int_U \phi \frac{\partial f_n}{\partial x_i} + 0 \quad (\text{since } Tf_n = 0)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_U \frac{\partial \phi}{\partial x_i} f = -\int_U \phi \frac{\partial f}{\partial x_i} \Longrightarrow \underbrace{\forall \phi \in C^\infty(\mathbb{R}^n) \int_{\partial U} Tf \, \phi \, \nu_i \, d\mathcal{H}^{n-1} = 0}$$

$\Rightarrow Tf = 0 \Rightarrow f \in W_0^{1,p}(U)$.

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ Banach spaces. $X \subseteq Y$ as a subspace

Def $X \hookrightarrow Y$ is a CONTINUOUS EMBEDDING
$f \longrightarrow f$

if $\exists\, C > 0$

$$\|f\|_Y \leq C \|f\|_X$$

Def A continuous embedding is COMPACT

$\forall$ sequence f_n in X such that $\exists\, C>0$ $\|f_n\|_X \leq C$

there exists f_{n_j} subsequence, $f \in Y$ such that

$$\|f_{n_j} - f\|_Y \longrightarrow 0$$

(bdd sequences in X are relatively compact in Y.

Ex

U bdd $\left(C^{0,\alpha}(U), \|\cdot\|_{0,\alpha}\right) \xhookrightarrow{\text{COMPACT}} \left(C(\overline{U}), \|\cdot\|_{\infty}\right)$

$$\|f\|_{C^{0,\alpha}} = \|f\|_{\infty} + \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$$

f_n bdd in $C^{0,\alpha}(U) \Rightarrow \|f_n\|_{\infty} \leq C \quad \sup_{x \neq y} \frac{|f_n(x)-f_n(y)|}{|x-y|^{\alpha}} \leq C.$

ASCOLI ARZELÀ $\Rightarrow f_{n_j} \to g$ uniformly $\quad g \in C(\overline{U})$

$\|f_{n_j} - g\|_{\infty} \to 0$

$$\left(C^{0,\alpha}(U), \|\cdot\|_{0,\alpha}\right) \hookrightarrow \left(C^{0,\beta}(U), \|\cdot\|_{0,\beta}\right) \quad \beta > \alpha$$

COMPACT for $\beta > \alpha$! (see ex 1 moodle)

Observ. $(X, \|\cdot\|_X) \xrightarrow{\text{CONTINUOUS}} (Y, \|\cdot\|_Y) \xrightarrow{\text{COMPACT}} (Z, \|\cdot\|_Z) \quad X \subseteq Y \subseteq Z$

COMPOSITION IS COMPACT

bdd sequences in X are bdd sequ. in Y and so relat. compact in Z.

What we proved for Sobolev spaces in DIMENS 1.

$f \in W^{1,p}(a,b)$ has a continuous representative s. that

$$f(x) = f(y) + \int_y^x f'(t)\,dt \quad \forall\, x,y \in [a,b]$$

[where $f'(t)$ is the weak derivative of f which also coincide with the derivative almost everywhere.

(not always true that derivative almost everywhere is the weak derivative!)

$p > 1$

$$|f(x) - f(y)| = \left| \int_x^y f'(t)\,dt \right| \leq \int_a^b |f'(t)|\,|\chi_{(x,y)}(t)|\,dt \leq \text{Hölder}$$

$$\leq \|f'\|_{L^p(a,b)} \cdot |x-y|^{1-\frac{1}{p}} \quad \Rightarrow \quad f \in C^{0,1-\frac{1}{p}}(a,b)$$

I want to prove $\|f\|_\infty \le C\|f\|_{W^{1,p}}$.

take $\bar{x}\in[a,b]$ $\|f\|_\infty = |f(\bar{x})|$ ($f \in C[a,b]$)

$|f(\bar{x})| = |f(y) + \int_y^{\bar{x}} f'(t)\,dt|$ < if $\bar{x}>a$ take $y\in[a,\bar{x}]$ ①; if $\bar{x}=a$ take $y\in[a,b]$

① $\le |f(y)| + \int_y^{\bar{x}} |f'(t)|\,dt$ (if $\bar{x}=a$ $\le |f(y)| + \int_a^y |f'(t)|\,dt$ and same argument...)

INTEGRATE in $y\in(a,\bar{x})$

$$|f(\bar{x})|(\bar{x}-a) \le \int_a^{\bar{x}} |f(y)|\,dy + \int_a^{\bar{x}}\int_y^{\bar{x}} |f'(t)|\,dt\,dy = \text{change order}$$

$\le \|f\|_\infty(b-a)$

$$\le \int_a^{\bar{x}} |f(y)|\,dy + \int_a^{\bar{x}} |f'(t)| \underbrace{\int_a^t dy}_{\le}\,dt =$$

$t-a \le \bar{x}-a \le b-a$

Holder

$$\le |\bar{x}-a|^{1-\frac{1}{p}}\|f\|_{L^p} + (\bar{x}-a)\cdot\|f'\|_{L^p}|\bar{x}-a|^{1-\frac{1}{p}}$$

$$\Rightarrow \|f\|_\infty \le \frac{\|f\|_{L^p}}{|b-a|^{1/p}} + \|f'\|_{L^p}|b-a|^{1-1/p} \le C_p\|f\|_{W^{1,p}(a,b)}.$$

therefore $W^{1,p}(a,b), \|\cdot\|_{1,p} \hookrightarrow C[a,b], \|\cdot\|_\infty$ CONTINUOUS $\forall p \in [1,+\infty]$

for $p>1$ $W^{1,p}(a,b), \|\cdot\|_{1,p} \hookrightarrow C^{0,1-\frac{1}{p}}(a,b), \|\cdot\|_{0,1-\frac{1}{p}}$ CONTINUOUS

since $\dfrac{|f(x)-f(y)|}{|x-y|^{1-1/p}} \le \|f'\|_{L^p}$

for $p>1$ $W^{1,p}(a,b)\ \|\cdot\|_{1,p} \hookrightarrow C[a,b], \|\cdot\|_\infty$ COMPACT

$$W^{1,p}(a,b) \xrightarrow{\text{contin.}} C^{0,1-\frac{1}{p}}(a,b) \xrightarrow{\text{compact}} C[a,b]$$

$W^{1,p}(a,b), \|\cdot\|_{1,p} \hookrightarrow C^{0,\alpha}(a,b)$ $\forall \alpha < 1-\frac{1}{p}$ COMPACT

$$W^{1,p}(a,b) \to C^{0,1-\frac{1}{p}}(a,b) \to C^{0,\alpha}(a,b)$$

Ex 1 sheet 4 on Moodle

$W^{1,p}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is CONTINUOUS

(there exists $C>0$
$\|f\|_\infty \leq C \|f\|_{W^{1,p}(\mathbb{R})}$)

but the embedding is <u>NOT COMPACT</u>

ex: $f_n(x) = \phi(x+n)$ $\phi \in C_c^\infty(\mathbb{R})$

$\|f_n\|_{W^{1,p}(\mathbb{R})} = \|\phi\|_{L^p} + \|\phi'\|_{L^p} = C$ fixed constant

$f_n(x) \to 0$ POINTWISE as $n \to +\infty$

BUT $f_n \not\to 0$ in $L^\infty(\mathbb{R})$ $\|f_n\|_\infty = \|\phi\|_\infty \neq 0$.

$f_n \not\to 0$ in $L^q(\mathbb{R})$ $\forall q \in [1, +\infty]$.