$C^\infty(U) \cap W^{k,p}(U)$ are dense in $W^{k,p}(U)$.

What about $C^\infty(\bar{U})$? We need to control geometrically the boundary

EXTENSIONS

Obs $u \in W^{1,p}(U) \to \bar{u} = \begin{cases} u & \text{in } U \\ 0 & \text{elsewhere} \end{cases}$

$\bar{u} \in L^p(\mathbb{R}^n)$ but $\bar{u} \notin W^{1,p}(\mathbb{R}^n)$

Theorem

Let $p \in [1, +\infty]$ U bdd of class C^1. (or $U = \mathbb{R}^n_+ = \{(x_1 \dots x_n), x_n > 0\}$)

Let $V \supset\supset U$ open bdd. Then $\exists\, E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ linear bdd operator

1) $Eu = u$ a.e in U

2) $\operatorname{supp} Eu \subset V$

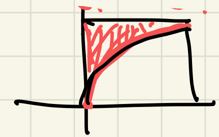
3) $\| Eu \|_{W^{1,p}(\mathbb{R}^n)} \leq C(p, U, V) \| u \|_{W^{1,p}(U)}$

(theorem true also for $p = +\infty$ — we come back to this case)

Obs if U has Lipschitz boundary, extension theorem works.

If U has Hölder boundary in general not true!

ex $\quad Q = (0,1) \times (0,1) \subseteq \mathbb{R}^2$



$$U = \{(x,y) \in Q \quad y > x^{1/3}\}$$

$$u_\gamma(x,y) = y^{-\gamma} \in W^{1,p}(U) \iff \int_0^1 \int_{x^{1/3}}^1 \frac{1}{y^{\gamma p}}\, dy < +\infty$$

$$\int_0^1 \int_{x^{1/3}}^1 \frac{1}{y^{(\gamma+1)p}}\, dy < +\infty$$

$p(\gamma+1) < 1+3 = 4$

take $p = 3 > 2 = \dim U$

$$\gamma < \frac{4}{3} - 1 = \frac{1}{3}$$

So for $\gamma < \frac{1}{3}$, $p = 3$ $\Rightarrow$ u_γ cannot be extended to $W^{1,3}(\mathbb{R}^2)$

since $W^{1,3}(\mathbb{R}^2) \subseteq L^\infty(\mathbb{R}^2) \quad \subseteq C^{0,\frac{2}{3}}(\mathbb{R}^2)$

(we will prove it)

Corollary [U bdd open, of class C^1, then $C^\infty(\bar U)$ is dense in $W^{1,p}(U)$ for $p \in [1,+\infty)$.

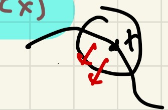
proof $E: W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$ with compact support

$C^\infty_c(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n) \Longrightarrow C^\infty_c(\mathbb{R}^n)|_U$ is dense in $W^{1,p}(U)$

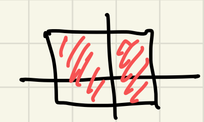
$C^\infty_c(\mathbb{R}^n)|_U \subseteq C^\infty(\bar U)$

Actually there exists a more direct proof of the previous result

U open bdd, in order that $C^\infty_c(\mathbb{R}^n)|_U$ is dense in $W^{1,p}(U)$ it is sufficient that [U has the SEGMENT PROPERTY: $\forall x \in \partial U$ $\exists r_x > 0$, $y_x \in \mathbb{R}^n$ such that $\forall z \in \bar U \cap B(x, r_x)$ $z + t y_x \in U$ $t \in (0,1)$



Counterex. to density $U = \{(x,y) \in \mathbb{R}^2 \quad 0 < |x| < 1 \quad |y| < 1\}$



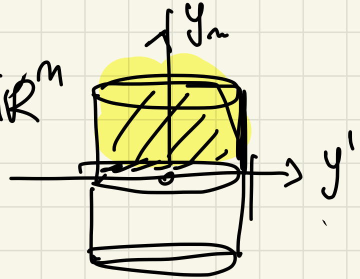
$f(x,y) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases} \Rightarrow \nexists\, \phi \in C^\infty(\bar U) \quad \|\phi - f\|_{W^{1,p}} < \varepsilon$

LEMMA (Extension by reflection)

$\delta > 0$.

$Q_\delta = \{(y', y_n) \subseteq \mathbb{R}^{n-1} \times \mathbb{R} \quad |y'| < \delta, |y_n| < \delta\} \subseteq \mathbb{R}^n$

$Q_\delta^+ = Q_\delta \cap \{y_n > 0\}$



Let $u \in W^{1,p}(Q_\delta^+)$ for $p \in [1, +\infty]$.

Define $u^* := u^*(y', y_n) = \begin{cases} u(y', y_n) & y_n > 0 \\ u(y', -y_n) & y_n < 0 \end{cases}$

Then $u^* \in W^{1,p}(Q_\delta)$, $\|u^*\|_{W^{1,p}(Q_\delta)} = 2\|u\|_{W^{1,p}(Q_\delta^+)}$

proof $\|u^*\|_{L^p(Q_\delta)} = 2\|u\|_{L^p(Q_\delta^+)}$ simple

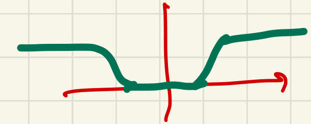
we prove that

① $\dfrac{\partial u^*}{\partial y_i} = \begin{cases} \dfrac{\partial u}{\partial y_i}(y', y_n) & y_n > 0 \\ \dfrac{\partial u}{\partial y_i}(y', -y_n) & y_n < 0 \end{cases}$

② $\dfrac{\partial u^*}{\partial y_n} = \begin{cases} \dfrac{\partial u}{\partial y_n}(y', y_n) & y_n > 0 \\ -\dfrac{\partial u}{\partial y_n}(y', -y_n) & y_n < 0 \end{cases}$

and so conclusion is obtained.

1 $\phi \in C_c^\infty(Q_\delta)$ $\displaystyle\int_{Q_\delta} u^* \frac{\partial \phi}{\partial x_i} = \int_{Q_\delta^+} u \frac{\partial \psi}{\partial x_i}$ $(*)$



$\psi(y', y_n) = \varphi(y', y_n) + \varphi(y', -y_n)$

Fix $\xi : \mathbb{R} \to [0,1]$, $\xi \in C^\infty(\mathbb{R})$, $\xi(t) = \begin{cases} 0 & |t| < \frac{1}{2} \\ 1 & |t| > 1 \end{cases}$

$\psi \notin C_c^\infty(Q_\delta^+)$! $\rightsquigarrow$ $\underline{\psi \cdot \xi(k y_n)} \in C_c^\infty(Q_\delta^+)$ (also in $C_c^\infty(Q_\delta^-)$)

$$\frac{\partial}{\partial x_i} \psi\, \xi(k y_n) = \xi(k y_n) \frac{\partial}{\partial x_i} \psi$$

$$\underbrace{\int_{Q_\delta^+} \xi(k y_n) \frac{\partial}{\partial x_i} \psi \cdot u}_{\downarrow\, k \to +\infty} = \underbrace{-\int_{Q_\delta^+} \xi(k y_n) \frac{\partial}{\partial x_i} u \cdot \psi}_{k \to +\infty}$$

by dominated convergence

$$\int_{Q_\delta^+} \frac{\partial}{\partial x_i} \psi\, u = -\int_{Q_\delta^+} \frac{\partial}{\partial x_i} u\, \psi$$

$(*) \Rightarrow \displaystyle\int_{Q_\delta} u^* \frac{\partial \phi}{\partial x_i} = -\int_{Q_\delta^+} \frac{\partial}{\partial x_i} u\, \psi = -\int_{Q} \frac{\partial u^*}{\partial x_i} \cdot \phi$

1 is proved

②

$$\int_{Q_\delta} u^* \frac{\partial \phi}{\partial y_n} = \int_{Q_\delta^+} u \cdot \frac{\partial \chi}{\partial y_n} \qquad \chi(y', y_n) = \phi(y', y_n) - \phi(y', -y_n)$$

$$\chi(y', 0) = 0 \longrightarrow |\chi(y', y_n)| \leq C|y_n| \quad \forall (y', y_n) \in Q_\delta$$

ξ as before $\quad \chi(y', y_n) \cdot \xi(k y_n) \in C_c^\infty(Q_\delta^+)$

$$\int_{Q_\delta^+} u \frac{\partial}{\partial y_n} (\xi(k y_n) \chi) = -\int_{Q_\delta^-} \frac{\partial u}{\partial y_n} \cdot \xi(k y_n) \chi \, dy$$

$$\frac{\partial}{\partial y_n} (\chi \xi(y_n k)) = \xi(k y_n) \frac{\partial}{\partial y_n} \chi + \underbrace{k \xi'(k y_n) \chi}$$

$$\left| \int_{Q_\delta^+} u \, k \xi'(k y_n) \chi(y) \right| \leq \int |u| \cdot \underbrace{k}_{} \|\xi'\|_\infty \cdot C \underbrace{|y_n|}_{} \, dy \leq$$

$\xi'(t) \equiv 0 \quad |t| < \frac{1}{2}, \quad |t| > 1 \qquad 0 < y_n < \frac{1}{k}$

$k|y_n| < 1$!

$$\leq C \|\xi'\|_\infty \int_{0 < y_n < \frac{1}{k}} |u| \cdot 1 \longrightarrow 0 \quad \text{as } k \to +\infty$$

Observation: Let $f \in W^{2,p}(Q_\delta^+)$.

$$f^*(y', y_n) = \begin{cases} f(y', y_n) & y_n > 0 \\ 4f\left(y', -\frac{y_n}{2}\right) - 3f(y', -y_n) & y_n < 0 \end{cases}$$

$\Rightarrow f^* \in W^{2,p}(Q_\delta)$

and $\|f^*\|_{W^{2,p}(Q_\delta)} \leq C \cdot \|f\|_{W^{2,p}(Q_\delta^+)}$

With similar arguments -