Let us fix (X, Σ, μ) measure space

X set $\quad \Sigma$ is a σ-algebra on X

$\mu : \Sigma \to [0, +\infty]$ measure

DEFINITION

$f : X \to \mathbb{R}^n$ is measurable
Σ, μ $\quad \mathcal{B}(\mathbb{R}^n)$ $\quad (f : X \to \mathbb{C} \simeq \mathbb{R}^2)$

is if the preimage of every Borel set is an element of Σ

$\forall A \in \mathcal{B}(\mathbb{R}^n) \quad f^{-1}(A) = \{x \in X \mid f(x) \in A\}$ = preimage of A

$\Rightarrow f^{-1}(A) \in \Sigma$.

in particular $f : X \to \mathbb{R}$ is measurable

if $\forall a \in \mathbb{R}$ $\underbrace{\{x \mid f(x) \leq a\}}_{f^{-1}} \in \Sigma$

$$f^{-1}(-\infty, a] \in \Sigma$$

$$A = (-\infty, a] \in \mathcal{B}(\mathbb{R})$$

(Actually it is sufficient to check that $\forall a \in \mathbb{R}$ $f^{-1}(-\infty, a] \in \Sigma$ to get that $\forall A \in \mathcal{B}(\mathbb{R}) \Rightarrow f^{-1}(A) \in \Sigma \Rightarrow f$ is measurable)

Ex (X, Σ, μ) is a probability space

(X set

Σ $\rightarrow$ σ-algebra of events

μ is a probability measure on X

$\Rightarrow$ this means $\boxed{\mu(X) = 1}$

$(X, \Sigma, \mu) \rightsquigarrow \boxed{(\Omega, \mathcal{F}, \mathbb{P})}$

$$\mathbb{X} : \Omega \to \mathbb{R}, \quad \omega \mapsto \mathbb{X}(\omega)$$ is a random variable

if it is a measurable function.

$\Rightarrow$ $\forall a \in \mathbb{R}$ $\{\omega \in \Omega \mid \mathbb{X}(\omega) \leq a\} \in \mathcal{F}$

Obs $f : (X, \Sigma, \mu) \longrightarrow \mathbb{R}^n$ measurable function

$\downarrow$

then f INDUCES on $\mathbb{R}$ a Borel measure called the push-forward of the measure μ.

$f_\# \mu \longrightarrow$ this is a Borel measure on $\mathbb{R}$

$\forall A \in \mathcal{B}(\mathbb{R}^n)$ $\quad f_\# \mu (A) = \mu \left(\underline{f^{-1}(A)} \right)$

$$f^{-1}(A) = \{ x \in X \mid f(x) \in A \} \in \Sigma$$

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R}$ be a RANDOM <u>variable</u> (measurable function)

the push forward of $\mathbb{P}$ with respect to X is a <u>Borel measure</u> on $\mathbb{R}$

$\forall A \in \mathcal{B}(\mathbb{R})$ $\quad \mathbb{P}_X(A) = \mathbb{P}\{\omega \in \Omega \mid X(\omega) \in A\}$

$$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}\{\omega \in \Omega \mid X(\omega) \in \mathbb{R}\} = \mathbb{P}(\Omega) = 1$$

$\mathbb{P}_X$ is a FINITE Borel measure on $\mathbb{R}$

↳ cumulative distr. funct $F(a) = \mathbb{P}_X(-\infty, a] =$

$= \mathbb{P}\{\omega \mid X(\omega) \leq a\}$

Integration with respect to a Borel finite or σ-finite measure on $\mathbb{R}$

$\boxed{(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)}$

μ Borel measure,

$\mu(\mathbb{R}) < +\infty$ FINITE

$\mathbb{R} = \bigcup_{n=1}^{\infty} C_n \quad \mu(C_n) < +\infty$

σ-finite

$\mathcal{M} = \overline{\mathcal{B}(\mathbb{R})}^{\mu}$ = completion of the Borel σ-algebra with respect to μ

$\mathcal{M} = \{ A \cup B, \quad A \in \mathcal{B}(\mathbb{R}) \quad B \subseteq C \quad \exists C \in \mathcal{B}(\mathbb{R}) \quad \mu(C) = 0 \}.$

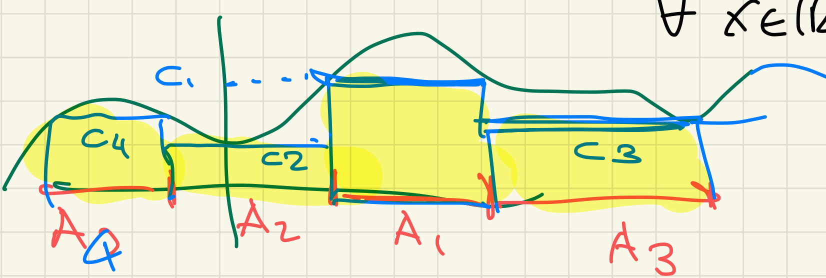
$f : (\mathbb{R}, \mathcal{M}, \mu) \longrightarrow \underline{\underline{\mathbb{R}}}$ measurable

$\forall A \in \mathcal{B}(\mathbb{R}) \rightarrow f^{-1}(A) \in \mathcal{M}$

$\boxed{\text{ASSUME}}$

$f \geq 0 \qquad f(x) \geq 0 \quad \forall x \in \mathbb{R}$

$$\int_{\mathbb{R}} f(x)\, d\mu = \sup \Big\{ \underbrace{\sum_{i=1}^{n} c_i \, \mu(A_i)}, \quad \boxed{\forall x \in \mathbb{R} : f(x) \geq \sum_{i=1}^{n} c_i \, \chi_{A_i}(x)} \Big\} \qquad \begin{matrix} c_i \geq 0 \\ A_i \in \mathcal{M} \end{matrix}$$



$$\chi_{A_i}(x) = \begin{cases} 1 & x \in A_i \\ 0 & x \notin A_i \end{cases}$$

If the $\sup\{ \sum_{1}^{n} c_i \, \mu(A_i)$ for all $c_i \geq 0$, $A_i \in \mathcal{M}$

such that

$$f(x) \geq \sum_{i=1}^{n} c_i \, \chi_{A_i}(x) \quad \forall x \in \mathbb{R} \} < +\infty$$

$\longrightarrow$ f is integrable

$$\int_{\mathbb{R}} f(x) \, d\mu = \sup \{ \; . \; . \; . \; \}$$

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f is NOT POSITIVE $\quad f = f^{+} - f^{-}$

$f^{+} = \max(f, 0) \geq 0 \quad f^{-} = \max(0, -f) \geq 0$

$$f^{+}(x) = \begin{cases} 0 & f(x) \leq 0 \\ f(x) & f(x) > 0 \end{cases} \qquad f^{-}(x) = \begin{cases} 0 & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

f is integrable if f^+ and f^- are both integrable

$$\int_{\mathbb{R}} f \, d\mu = \int_{\mathbb{R}} f^+ \, d\mu - \int_{\mathbb{R}} f^- \, d\mu .$$

If $\mu =$ Lebesgue this is the classical integral (coinciding with the Riemann integral if f is continuous)

μ_f $\quad \mu(A) = \# \{ n \in \mathbb{N} \quad n \in A \}$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 & 1 \leq x < 2 \\ 2 & 2 \leq x < 3 \\ \vdots & \end{cases}$$

$F(x) = [x] = \{$smallest natural number n

$\cancel{n} \leq x \}$

$f : (\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu) \longrightarrow \mathbb{R}$ meas. $\quad f \geq 0$

$$\int_{\mathbb{R}} f \, d\mu = \sum_{1}^{+\infty} f(i)$$

$\boxed{\text{Definition}}$:

We say μ, ν Borel measures on $\mathbb{R}$ are SINGULAR one with respect to the other

$$\mu \perp \nu \quad \text{if} \quad \mathbb{R} = A \cup \underline{\underline{B}} \qquad A \cap B = \emptyset$$

$$\mu(A) = 0 \qquad \nu(B) = 0$$

ex. $\mu = \mathcal{L}$ Lebesgue measure

δ_0 Dirac measure $\quad \delta_0(A) = \begin{cases} 1 & 0 \in A \\ 0 & 0 \notin A \end{cases}$

$$\mathcal{L} \perp \delta_0 \qquad \mathbb{R} = (\mathbb{R} \setminus \{0\}) \cup \{0\}$$

$$\underline{\mathcal{L}\{0\} = 0} \qquad \delta_0(\mathbb{R} \setminus \{0\}) = 0$$

Ex $\mu(A) = \#\{n \in \mathbb{N} \quad n \in A\}$

$\mathcal{L} \perp \mu \qquad \mathbb{R} = (\mathbb{R} \setminus \mathbb{N}) \cup \mathbb{N}.$

$\mu(\mathbb{R} \setminus \mathbb{N}) = 0$

$\mathcal{L}(\underline{\underline{\mathbb{N}}}) \underline{\underline{=}} \mathcal{L}\left(\bigcup_{i=0}^{+\infty} \{i\}\right) = \sum_{i=0}^{n} \mathcal{L}\{i\} = 0$

μ, ν are Borel measures on $\mathbb{R}$

$\boxed{\mu \ll \nu \quad (\mu \text{ is ABSOLUTELY CONTINUOUS with respect to } \nu)}$

if $\forall A \in \mathcal{B}(\mathbb{R})$ such that $\nu(A) = 0$

it holds that $\mu(A) = 0$

(I'm not asking that if $\mu(B) = 0$ then $\nu(B) = 0$)

Ex $\boxed{\nu = \mathcal{L} \text{ Lebesgue measure}}$

$f \geq 0$ $\quad f: \mathbb{R} \to \mathbb{R}$ measurable such that

$$\int_{\mathbb{R}} f(x)\, d\mathcal{L} = \int_{\mathbb{R}} f(x) \underbrace{dx}_{\text{integral with respect to } \mathcal{L}} < +\infty$$

$\forall A \in \mathcal{B}(\mathbb{R})$

$$\mu(A) := \int_{\mathbb{R}} \underbrace{f(x) \cdot \chi_A(x)}\, dx = \int_A f(x)\, dx \leq \int_{\mathbb{R}} f(x)\, dx < +\infty$$

$$\chi_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases} \qquad f(x)\chi_A(x) = \begin{cases} f(x) & x \in A \\ 0 & x \notin A \end{cases}$$

$\mu(A) = \int_A f(x)dx$ is ACTUALLY a

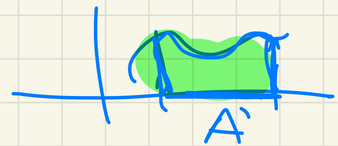
finite Borel measure and

$\left(\mu(\mathbb{R}) = \int_{\mathbb{R}} f(x)dx \right)$

$\mu \ll \mathcal{L}$ $\Longrightarrow$ $\boxed{\text{since if } \mathcal{L}(A) = 0 \Rightarrow \int_A f(x)dx = 0}$

$f \geq 0$ $\mathcal{L}$ = Lebesgue = length measure

$\int_A f(x)dx$ = area of the subgraph of f on A



Theorem (RADON - LEBESGUE - NIKODYM)

Let μ be a FINITE Borel measure on $\mathbb{R}$

(also for σ-finite true).

there exists [a unique Borel measure $\underline{\rho}$
a unique Borel measure ν]

such that $\mu(A) = \rho(A) + \nu(A) \quad \forall A \in \mathcal{B}(\mathbb{R})$

$\boxed{(\mu = \rho + \nu)}$.

$\rho \perp \mathcal{L}$ (ρ is singular with respect to Lebesgue)

$\nu \ll \mathcal{L}$ (ν is absolutely cont with respect to $\mathcal{L}$)

and moreover $\exists f \geq 0$ $f : \mathbb{R} \to \mathbb{R}$
measurable with respect to Lebesgue and
integrable such that

$$\boxed{\nu(A) = \int_A f(x)\,dx}$$

f is called the DENSITY of ν.

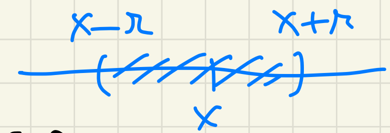
$\boxed{\mu \text{ Borel finite}} \to \underbrace{\exists\, \rho \perp \mathcal{L}}$ $\quad$ ρ FINITE

$\boxed{\exists f \geq 0 \text{ integrable}}$

$$\mu(A) = \underbrace{\rho(A)} + \underbrace{\int_A f(x)\,dx} \qquad \forall A \in \mathcal{B}(\mathbb{R})$$

the density f can be obtained as follows:

fix $x \in \mathbb{R}$



$$\lim_{r \to 0^+} \frac{\mu\left((x-r, x+r)\right)}{2r} = f(x)$$

$$2r = \mathcal{L}(x-r, x+r)$$

(one can show that for a.e. x this limits exists)

$\underline{X} : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathbb{R}$ random variable

$$\boxed{\underline{\mathbb{P}}_{\underline{X}} \quad \text{finite Borel measure on } \mathbb{R}.}$$

$\underline{\mathbb{P}}_{\underline{X}}(A) = \mathbb{P}\{\omega \in \Omega, \ \underline{X}(\omega) \in A\}$ $\qquad \underline{\mathbb{P}}_X(\mathbb{R}) = 1.$

we say that $\underline{X}$ is an absolutely continuous random variable if $\underline{\mathbb{P}}_X \ll \mathcal{L}$
(if $\mathbb{P}_X$ is absolutely continuous with respect to Lebesgue)

density

$\mathbb{P}_X \ll \mathcal{L}$ (so the singular part is 0)

- density $\forall x \in \mathbb{R}$

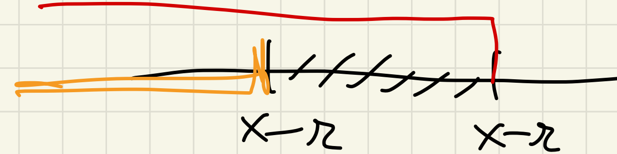
$$f(x) = \lim_{r \to 0^+} \frac{\mathbb{P}_X(x-r, x+r)}{2r} =$$

$$= \lim_{r \to 0^+} \frac{\mathbb{P}\{\omega \in \Omega, \ x-r < X(\omega) < x+r\}}{2r}$$

$\{\omega \in \Omega \ \ x-r < X(\omega) < x+r\} =$

$= \{\omega \in \Omega, \ \ X(\omega) < x+r\} \setminus \{\omega \in \Omega \mid X(\omega) \leq x-r\}$

$$\mathbb{P}\{\omega \mid x-r < X(\omega) < x+r\} =$$



$$= \mathbb{P}\{\omega \mid X(\omega) < x+r\} - \mathbb{P}\{\omega \mid X(\omega) \le x-r\}$$

$$= \mathbb{P}\{\omega \mid X(\omega) \le x+r\} - \mathbb{P}\{\omega \mid X(\omega) = x+r\}$$

$$- \mathbb{P}(\omega \mid X(\omega) \le x-r) =$$

$$\mathbb{P}_X << \mathcal{L}$$

$$= G(x+r) - G(x-r) - \underbrace{\cancel{\mathbb{P}_X(\{x+r\})}}_{} = 0$$

$$f(x) = \lim_{r \to 0^+} \frac{G(x+r) - G(x-r)}{2r} = \lim_{r \to 0^+} \frac{G(x+r) - G(x)}{2r} + \frac{G(x) - G(x-r)}{2r}$$

$$= \frac{1}{2} \lim_{r \to 0^+} \frac{G(x+r) - G(x)}{r} + \frac{1}{2} \lim_{r \to 0^+} \frac{G(x) - G(x-r)}{r}$$

if G is differentiable at x

$$\lim_{r \to 0^+} \frac{G(x+r) - G(x)}{r} = G'(x) = \lim_{r \to 0^+} \frac{G(x) - G(x-r)}{r}$$

$f(x) = G'(x)$ if G is differentiable at x.

(density of $\mathbb{P}_X$ at x is the derivative of the cumulative distribution function at x.).

A result in measure theory says that if

$G : \mathbb{R} \to \mathbb{R}$ is MONOTONE NON DECREASING

if $x > y$, $G(x) \geq G(y)$

then G is differentiable at almost every point (G is differentiable $\forall x \in A \in \mathcal{B}(\mathbb{R})$ s.t. $\mathcal{L}(\underline{\underline{\mathbb{R} \setminus A}}) = 0$).

$X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ random variable

$\mathbb{P}_X$ finite Borel measure

$G(x) =$ cumulative distn. $= \mathbb{P}_X(-\infty, x] = \mathbb{P}\{\omega \mid X(\omega) \leq x\}$

G is monotone non decreasing

$x < y \qquad (-\infty, x] \subseteq (-\infty, y]$

$$G(x) = \mathbb{P}_X(-\infty, x] \leq \mathbb{P}_X(-\infty, y] = G(y)$$

⊙ G is differentiable $\underline{\text{almost everywhere}}$

$x \in A \qquad \mathcal{L}(\mathbb{R} \setminus A) = 0$

$$\boxed{f(x) = G'(x) \quad \forall x \in A}$$

$f \geq 0$ density.

$\mathbb{P}_X = \nu + \rho \qquad \underbrace{\nu(A) = \int f(x)\,dx}_{A} \qquad \nu \ll \mathcal{L}$

$\rho = ?$

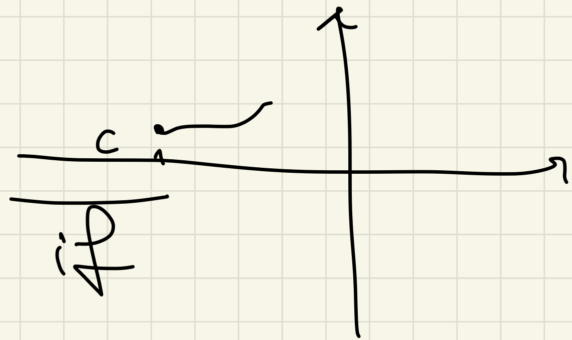
G is a monotone non decreasing function

(it is differentiable a.e).

↓

G has at most a **COUNTABLE** number of jumps.

↓



$c \in \mathbb{R}$ is a jump for G if

$$\boxed{G(c) - \lim_{x \to c^-} G(x) > 0}$$

$$G(c) - \lim_{x \to c^-} G(x) = \mathbb{P}_X\{c\} = \mathbb{P}\{\omega \mid X(\omega) = c\}$$

$\{c_i\}_{i \in \mathbb{N}}$ c_i jumps of G.

$G(c_i) - \lim_{x_i \to c_i^-} G(x) > 0$

$\mathbb{P}\{\omega \mid X(\omega) = c_i\} = \underline{\underline{\lambda_i > 0}}$

$\mathbb{P}_X\{c_i\}$

$$\mathbb{P}_X(A) = \underbrace{\rho(A)} + \int_A f(x)\,dx \qquad A \in \mathcal{B}(\mathbb{R})$$

$\rho \perp \mathcal{L}$

$$\boxed{\rho_d(A) = \sum_{\substack{i \in \mathbb{N} \\ c_i \in A}} \lambda_i}$$

discrete part

$$\rho_d = \sum_{i=1}^{+\infty} \lambda_i \, \delta_{\{c_i\}}$$

X random variable

$\downarrow$

$\mathbb{P}_X$ $\qquad$ G cumulative distribution function

$\{c_i\}$ = jumps of G

$$G(c_i) - \lim_{x \to c_i^-} G(x) = \lambda_i > 0$$

$$\boxed{\mathbb{P}_X(A) = \underbrace{\int_A G'(x)\,dx}_{} + \underbrace{\sum_{\substack{i \in \mathbb{N} \\ c_i \in A}} \lambda_i}_{} + \underbrace{\rho_s(A)}_{\rho_s \perp \mathcal{L}}}$$

X is an absolutely continuous r.v.

$$\mathbb{P}_X \ll \mathcal{L} \Rightarrow \mathbb{P}_X(A) = \int_A G'(x)\,dx .$$

NO JUMPS $\hookrightarrow$ $\boxed{G \text{ is CONTINUOUS}}$

NO SING. PART

$$\forall a \in \mathbb{R}$$

$$\mathbb{P}(\omega \mid X(\omega) = a) = 0$$

$$\mathbb{P}(\omega \mid X(\omega) \leq a) = \mathbb{P}(\omega \mid X(\omega) < a)$$

X is <u>discrete</u> if $G'(x) = 0$ a.e.

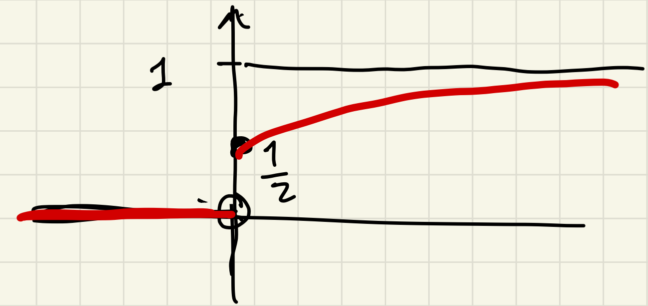
and $\mathbb{P}_X(A) = \sum_{c_i \in A} \lambda_i \qquad \mathbb{P}_X = \sum_i \lambda_i \delta_{c_i}$

G is CONSTANT A.E ~~and~~ then JUMPS at c_i.

Example

take

$$\underline{G(x)} = \begin{cases} 0 & x < 0 \\ \underbrace{1 - \frac{e^{-x}}{2}}_{} & x \geq 0 \end{cases}$$



$G(0) = \frac{1}{2}$

$$\underset{\text{density}}{\underline{\underline{G'(x)}}} = \begin{cases} 0 & \underline{\underline{x < 0}} \\ \frac{e^{-x}}{2} & \underline{x > 0} \end{cases}$$

$\Rightarrow$ the density is

$$f(x) = \frac{e^{-x/2}}{2} \cdot \chi_{(0,+\infty)}(x)$$

At $x = 0$ G is not differentiable

$c_1 = 0$ is a jump

$$G(0) - \lim_{x \to 0^-} G(x) = \frac{1}{2} - 0 = \frac{1}{2} = \lambda_1 > 0$$

$$\mathbb{P}_X(A) = \int_{A \cap \{x>0\}} \frac{e^{-x}}{2}\,dx + \frac{1}{2}\underbrace{\delta_0(A)}$$

$$\frac{1}{2} = G(0) - \lim_{x \to 0^-} G(x)$$

$$\mathbb{P}_X(\mathbb{R}) = \underbrace{\int_0^{+\infty} \frac{e^{-x}}{2}\,dx}_{\int_{\mathbb{R}\cap\{x>0\}}} + \frac{1}{2}\underbrace{\delta_0(\mathbb{R})} =$$

$$= \left[-\frac{e^{-x}}{2}\right]_0^{+\infty} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\mathbb{P}_X(-1,3) = \int_{(-1,3)\cap\{x>0\}} \frac{e^{-x}}{2}\,dx + \frac{1}{2}\delta_0(-1,3)$$

$$\mathbb{P}_X(A) = \int_{A\cap\{x>0\}} \frac{e^{-x}}{2}\,dx + \frac{1}{2}\delta_0(A) =$$

$\downarrow$

$\overline{X}$ as a random variable.

$$X = \frac{1}{2}\overline{X}_1 + \frac{1}{2}\overline{X}_2$$

$\overline{X}_1 \to$ Gamma random variable (exponential random v.).

$\overline{X}_2 \to$ Bernoulli r.v.