Most complete reference ADAMS-FOURNIER "Sobolev spaces"

$$W^{k,p}(U) = \{ u \in L^p(U) \quad \underbrace{D^\alpha u}_{\substack{\uparrow \\ \text{weak - derivative}}} \in L^p(U) \quad \forall\, |\alpha| \le k \}$$

$$\|u\|_{W^{k,p}} = \sum_{|\alpha| \le k} \|D^\alpha u\|_p = (\text{an equivalent norm for } p < +\infty)$$

$$\simeq \left[\sum_{|\alpha| \le k} \|D^\alpha u\|_p^p \right]^{1/p}.$$

- $W^{1,p}(a,b) \subseteq AC(a,b)$ $W^{1,1}(a,b) = AC((a,b))$

also $W^{1,p}(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ $W^{1,p}(a,b) \subsetneq AC(a,b)$

$\subseteq C^{0,1-\frac{1}{p}}(a,b)$

- in $U \subseteq \mathbb{R}^n$, $n > 1$ $W^{1,p}(U) \not\subseteq C(U)$ if $p \le n$

$$W^{1,p} \overset{\text{close}}{\subseteq} L^p \times (L^p \times \ldots \times L^p)$$

Def $W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_{k,p}} \subseteq W^{k,p}(U)$ CLOSED SUBSPACE

$W^{k,p}(U)$, $W_0^{k,p}(U)$ are reflexive for $p \in (1, +\infty)$.

Duals ?

Obs $W^{1,p}(U) \underset{\text{closed}}{\subseteq} \underbrace{L^p(U) \times [\underbrace{L^p(U) \times \dots \times L^p(U)}_{n \text{ times}}]}$ dual here is $(L^{p'}) \times \dots \times (L^{p'})$

$p' = \frac{p}{p-1}$

Obs $W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_{k,p}}$

So $(W_0^{k,p}(U))'$ is a space of distributions $(\subseteq \mathcal{D}'(U))$

(indeed if $T \in (W_0^{k,p}(U))' \Longrightarrow T\big|_{C_c^\infty(U)} \in \mathcal{D}'(U)$ and moreover

if $T = S$ on $\mathcal{D}'(U)$ $\underbrace{T(\phi) = S(\phi)\ \forall \phi \in C_c^\infty(U)}$ $\Rightarrow$ $T = S$ on $(W_0^{k,p}(U))'$

$(W^{k,p}(U))'$ is NOT A SPACE of distributions

(in the sense $T \in (W^{k,p}(U))' \Longrightarrow T_{|C_c^\infty(U)} \in \mathcal{D}'(U)$ but

this restriction does not identify T (there are different $S \in (W^{k,p}(U))'$ such that $T_{|C_c^\infty(U)} = S_{|C_c^\infty(U)}$.

$(W^{1,2}(0,1))'$ $\quad T : W^{1,2}(0,1) \to \mathbb{R}$ $\quad T_{|C_c^\infty(0,1)} \equiv 0$

$$u \longmapsto \int_0^1 u'(t)\,dt$$

more generally one consider $f_1 \dots f_n \in L^2(U)$ s. that $\sum_i \frac{\partial f_i}{\partial x_i} = 0$ in the sense of distributions

and $T(v) = \int_U \sum_i \frac{\partial f_i}{\partial x_i} \frac{\partial}{\partial x_i} v \, dx \not\equiv 0$ but $T \equiv 0$ as $\mathcal{D}'(U)$

Consider the case $p=2$, so $W^{k,2}(U)$ is Hilbert space

take $k=1$

by Riesz

$T \in W_0^{1,2}(U)' \iff \exists f \in W_0^{1,2}(U)$ s. that $\forall g \in W_0^{1,2}(U)$

$T(g) = \int_U f \cdot g + \sum_i \int_U \frac{\partial}{\partial x_i} f \cdot \frac{\partial}{\partial x_i} g$

$T \in W^{1,2}(U)' \iff \exists f \in W^{1,2}(U)$ s. that $\forall g \in W^{1,2}(U)$

$T(g) = \int_U f \cdot g + \sum_i \int_U \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}$.

The fact is that in general it is better not to identify a Hilbert space and its dual.

If $V \subseteq H$ Hilbert then $V' \supseteq H'$ (but $H \simeq H'$, $V \simeq V'$)...

$$W_0^{1,2}(U) \subseteq L^2(U) \subseteq \left(W_0^{1,2}(U)\right)'$$

↑ obvious

(I want to characterize $(W_0^{1,2}(U))'$ as particular distributions..

$(W_0^{1,2}(U))'$ is a Banach space contained in $D'(U)$

Prop Let $T \in D'(U)$. then $T \in (W_0^{1,2}(U))'$ if and only if $\exists f_0, f_1, \dots f_n \in L^2(U)$ such that

$$T = T_{f_0} - \sum_i \frac{\partial}{\partial x_i} T_{f_i}$$

(that is $T(\phi) = \int_U f_0 \phi + \sum_{i=1}^{n} \int_U f_i \frac{\partial \phi}{\partial x_i} dx$ $\forall \phi \in C_c^\infty(U)$)

proof

$\Rightarrow$ $T \in (H_0^1(U))' \Rightarrow \overset{\text{Riesz}}{\exists} u \in H_0^1(U)$ s. that $T(v) = \int_U u v \, dx + \sum_i \int_U \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}$

$f_0 = u \quad f_i = + \frac{\partial u}{\partial x_i}$

$\Leftarrow$ T is a distribution like this

$$|T(\phi)| \underset{\text{Holder}}{\leq} \|f_0\|_2 \|\phi\|_2 + \sum_i \|f_i\|_2 \|\phi_{x_i}\|_{L^2} \leq$$

$$\leq \left[\|f_0\|_2 + \sum_i \|f_i\|_2\right] \|\phi\|_{W^{1,2}} \qquad \forall \phi \in C_c^\infty(U)$$

$\rightarrow$ T can be extended to a linear cont functional on $H_0^1(U) = W_0^{1,2}(U)$. with norm $\|T\| = \|f_0\|_2 + [\sum_i \|f_i\|_2]$.

Obs By Riesz theorem in Hilbert spaces it is also true that

$T \in (W^{1,2}(U))' \implies \exists g \in W^{1,2}(U)$ such that

$$T(h) = \int_U h g \, dx + \sum_i \int_U \frac{\partial h_i}{\partial x_i} \frac{\partial g}{\partial x_i}.$$

as a distribution

$$\boxed{\text{so } T = T_g + \sum_i T_{\frac{\partial}{\partial x_i} g} = T_g + \sum_i \frac{\partial}{\partial x_i} T_g}$$

but NOT all $T \in (W^{1,2}(U))'$ corresponds to distributions written as $T_{g_0} + \sum_i \frac{\partial}{\partial x_i} T_{g_i}$ $g_0, g_i \in L^2$.

(there are $T \in (W^{1,2}(U))'$ which are 0 if computed on $C_c^\infty(U)$.)

DENSITY RESULTS (density of smooth functions)

Lemma $f \in W^{k,p}(U)$ $p \in [1, +\infty)$

then $\forall U' \subset\subset U$ (U' open bdd, $\overline{U'} \subseteq U$)

$f * \eta_\varepsilon \in C^\infty(U')$ for ε suff. small and $f * \eta_\varepsilon \to f$ in $W^{k,p}(U')$

This means that $f * \eta_\varepsilon \to f$ in $W^{k,p}_{loc}(U)$ as $\varepsilon \to 0$.

proof. Fix U' $\varepsilon_0 = \operatorname{dist}(U', \mathbb{R}^n \setminus U) > 0$

$\forall \varepsilon < \varepsilon_0$ $f * \eta_\varepsilon(x) = \int_{B(0,\varepsilon)} f(x-y)\, \eta_\varepsilon(y)\, dy \in C^\infty(U')$ $(x - y \in U)$

$f * \eta_\varepsilon \to f$ in $L^p(U')$. We compute $D^\alpha(f * \eta_\varepsilon)$ in weak sense: $\phi \in C_c^\infty(U')$

$$(-1)^{|\alpha|} \int_{U'} D^\alpha \phi \cdot f * \eta_\varepsilon \, dx = (-1)^{|\alpha|} \int_{U'} D^\alpha \phi(x) \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\, f(x-y)\, dy\, dx =$$

$$= (-1)^{|\alpha|} \int_{B(0,\varepsilon)} \eta_\varepsilon(y) \int_{U'} D^\alpha \phi(x) f(x-y)\, dx = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) \int_{U'} D^\alpha f(x-y) \phi(x)\, dx = \int_{U'} (D^\alpha f * \eta_\varepsilon)\, \phi$$

$\to (-1)^{|\alpha|}\int_{U'} D^\alpha\phi \; f*\eta_\varepsilon(x)\,dx = \int_{U'} \phi\,(D^\alpha f)*\eta_\varepsilon\,dx$

$\Rightarrow (D^\alpha f)*\eta_\varepsilon \longrightarrow D^\alpha f$ in $L^p(U')$ $\square$.

Theorem (MEYERS-SERRIN 1964) $U\subseteq\mathbb{R}^n$ open

$\left[\mathcal{C}^\infty(U)\cap W^{k,p}(U)\right]$ is dense in $W^{k,p}(U)$, $p\in[1,+\infty)$

Obs 1 Not $\mathcal{C}^\infty(\overline{U})$!

Obs 2 $\overline{\mathcal{C}^\infty(U)\cap W^{k,\infty}(U)}^{\|\cdot\|_{k,\infty}} = \mathcal{C}^k(U)\cap W^{k,\infty}(U)$

ex $\phi(x)=|x|\in W^{1,\infty}(-1,1)$ but $\nexists\,\phi_n\in\mathcal{C}^\infty(-1,1)$

$\phi_n\longrightarrow\phi$ in $\|\cdot\|_{W^{1,\infty}}$

let $\phi_n\to\phi$ ϕ_n is Cauchy $\Rightarrow \|\phi_n-\phi_m\|_\infty\to 0$ $\|\phi_n'-\phi_m'\|_\infty\to 0$ $\Rightarrow$ $\phi_n\to\phi$ uniformly, $\phi_n'\to\phi'$ uniformly

Corollary $W_0^{k,p}(\mathbb{R}^n)=W^{k,p}(\mathbb{R}^n)=\overline{\mathcal{C}_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{k,p}}$

$\to$ indeed $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{C}^\infty(\mathbb{R}^n)\cap W^{k,p}(\mathbb{R}^n)$

$\phi=\begin{cases}1 & |x|\le 1\\ 0 & |x|>2\end{cases}$ $0\le\phi\le 1$ $\quad\psi\in\mathcal{C}^\infty(\mathbb{R}^n)\cap W^{k,p}(\mathbb{R}^n)$

$\phi\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ $\quad\mathcal{C}_c^\infty(\mathbb{R}^n)\ni\psi_r=\psi\cdot\phi\left(\frac{x}{r}\right)\longrightarrow\psi$ in $W^{k,p}(\mathbb{R}^n)$

CUT-OFF FUNCTION $\quad r\to+\infty$

proof of Meyers Serrin. $U \subseteq \mathbb{R}^n$

$U_k = \{ x \in U \quad |x| < k \quad \operatorname{dist}(x, \mathbb{R}^n \setminus U) > \frac{1}{k} \}$ $\qquad U_0 = \emptyset$

$V_k = U_{k+1} \setminus \overline{U_{k-1}} = \{ x \in U, \ k-1 < |x| < k+1 \quad \frac{1}{k+1} < \operatorname{dist}(x, \mathbb{R}^n \setminus U) < \frac{1}{k-1} \}$

$\overline{V_k} \subset U$ compact.
$U = \bigcup_{k=1}^{\infty} V_k$

] ξ_k partition of unit
$\xi_k \in C_c^\infty(U) \quad 0 \le \xi_k \le 1 \quad \operatorname{supp} \xi_k \subseteq V_k$

$\forall A \subset\subset U \quad \exists I_A \ \# I_A < +\infty \quad \sum_{k \in I_A} \xi_k(x) = 1.$

(for $A = \overline{V_k} \Rightarrow \xi_i \equiv 0 \quad i \ge k+2$

Let $f \in W^{r,p}(U)$ By lemma $(f \cdot \xi_k) * \eta_\varepsilon \longrightarrow f \xi_k$ in $W^{r,p}(U_{k+2} \setminus \overline{U_{k-2}})$

Fix $\delta > 0$ and find $\varepsilon_k < \frac{1}{(k+1)(k+2)}$ = dist $((U_{k+2} \setminus \overline{U_{k-2}}), \mathbb{R}^n \setminus U)$ (so $\varepsilon_k \to 0$ as $k \to +\infty$)

such that $\| f \xi_k * \eta_{\varepsilon_k} - f \xi_k \|_{W^{r,p}} \le \delta / 2^{k+1}$

ε_k depends on δ!

Note that $f = \sum_k f \xi_k$ (NOT INFINITE SUM!

$$\Phi = \sum_k \left[f \xi_k * \eta_{\varepsilon_k} \right] \in C^\infty(U)$$

the sum is finite

Let $V_{k_0} \subset\subset U \Rightarrow \|\phi - f\|_{W^{r,p}(V_{k_0})} \leq \sum_{k \leq k_0+2} \| f\xi_k * \eta_{\varepsilon_k} - f\xi_k \|_{W^{r,p}(U')}$

$$\leq \sum_{k \leq k_0+2} \frac{\delta}{2^{k+1}} \leq \sum_k \frac{\delta}{2^{k+1}} = \delta$$

So $\forall \delta > 0 \ \exists \Phi_\delta \in C^\infty(U)$ depending on δ such that
$\|\phi - f\|_{W^{r,p}(U')} \leq \delta \quad \forall U' \subset\subset U$

conclude by taking the supremum on U'

or by monotone converging theorem $\square$