COMPUTABILITY (11/11/2024)

* DIAGONALISATION

idea : x_i $i \in I$

x_0 x_1 x_2 . . . x_i . . . .

↑ position i inside x_i

aim : build x of the same nature of the x_i's

different from all of them ($x \neq x_i \ \forall i$)

by letting $x \neq x_i$ "at position i"

Cantor $\forall X$ set

$|X| < |2^X|$ $2^X = \{ Y \mid Y \subseteq X \}$

when X is finite e.g. $X = \{0,1\}$ $2^X = \{\emptyset, \{0\}, \{1\}, X\}$

$|X| = 2 < 4 = |2^X| = 2^{|X|}$

Example : $|\mathbb{N}| < |2^{\mathbb{N}}|$

proof

assume $|\mathbb{N}| \geq |2^{\mathbb{N}}|$ i.e. there is $f: \mathbb{N} \to 2^{\mathbb{N}}$ surjective

($2^{\mathbb{N}}$ countable)

$\overbrace{\qquad\qquad\qquad\qquad}^{2^{\mathbb{N}}}$

	X_0	X_1	X_2	- - - -
0	~~YES~~ NO	NO	NO	
1	NO	~~NO~~ YES	NO	
2	YES	YES	~~YES~~ NO	
⋮	NO	YES	NO	
	NO	NO	NO	
	⋮	⋮		

$X_0 = \{0, 2\}$

$D = \{ i \mid i \notin X_i \}$

$\subseteq \mathbb{N}$

there is $K \in \mathbb{N}$ s.t. $D = X_K$

two possibilities

- $K \in D \Rightarrow K \notin X_K = D$ contradiction
- $K \notin D \Rightarrow K \in X_K = D$ "

$\Big\}$ absurd

$\Rightarrow 2^{\mathbb{N}}$ is not countable. $\square$

<u>EXERCISE</u>: $\mathcal{F} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$

$|\mathcal{F}| > |\mathbb{N}|$

(1st solution)

$$\mathcal{F}_2 = \left\{ f \in \mathcal{F} \;\middle|\; \begin{array}{l} f : \mathbb{N} \to \mathbb{N} \text{ total} \\ img(f) = \{0,1\} \end{array} \right\} \subseteq \mathcal{F}$$

$\mathcal{F}_2 \subseteq \mathcal{F}$

$\mathcal{F}_2 \to \mathcal{F}$

$f \mapsto f$

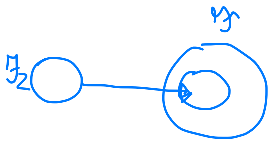
$\leadsto$

$$\begin{array}{ccc} |\mathcal{F}_2| & = & |2^{\mathbb{N}}| \\ \wedge\!| & & \vee \\ |\mathcal{F}| & & |\mathbb{N}| \end{array}$$

bijection

$\mathcal{F}_2 \to 2^{\mathbb{N}}$

$f \mapsto f^{-1}(1)$
$= \{ n \mid f(n) = 1 \}$



(2nd solution) $|\mathcal{F}| > |\mathbb{N}|$

consider an enumeration of elements in $\mathcal{F}$

	f_0	f_1	f_2	f_3	----
0	$f_0(0)$	$f_1(0)$	$f_2(0)$	----	
1	$f_0(1)$	$f_1(1)$	$f_2(1)$	---	
2	$f_0(2)$	$f_1(2)$	$f_2(2)$	---	
⋮	⋮	⋮	⋮		

$\downarrow +1$ $\quad \uparrow \to 0$

$f : \mathbb{N} \to \mathbb{N} \in \mathcal{F}$

$$f(n) = \begin{cases} f_n(n) + 1 & \text{if } f_n(n)\downarrow \\ 0 & \text{if } f_n(n)\uparrow \end{cases}$$

we have $f \neq f_n \quad \forall n \in \mathbb{N}$ since $f(n) \neq f_n(n)$ by construction

$\Rightarrow$ there is no enumeration which includes the entire $\mathcal{F}$ $\Rightarrow |\mathcal{F}| > |\mathbb{N}|$ $\square$

<u>OBSERVATION</u>: The following function is total and not computable

$f : \mathbb{N} \to \mathbb{N}$

$$f(n) = \begin{cases} \varphi_n(n) + 1 & \text{if } \varphi_n(n)\downarrow \quad [n \in W_n] \\ 0 & \text{if } \varphi_n(n)\uparrow \quad [n \notin W_n] \end{cases}$$

$$\begin{array}{c|cccc} & \varphi_0 & \varphi_1 & \varphi_2 & \cdots \\ \hline 0 & \varphi_0(0) & \varphi_1(0) & \varphi_2(0) & \cdots \\ 1 & \varphi_0(1) & \varphi_1(1) & \varphi_2(1) & \cdots \\ 2 & \varphi_0(2) & \varphi_1(2) & \varphi_2(2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

→ f is total by construction

→ f is not computable since $\forall n \quad f \neq \varphi_n$

$\forall n \in \mathbb{N} \qquad f(n) \neq \varphi_n(n)$

- $\varphi_n(n)\downarrow \quad \Rightarrow \quad f(n) = \varphi_n(n) + 1 \neq \varphi_n(n)$
- $\varphi_n(n)\uparrow \quad \Rightarrow \quad f(n) = 0 \neq \varphi_n(n)$

<u>EXERCISE</u>: Let $f : \mathbb{N} \to \mathbb{N}$ be a fixed function, $m \in \mathbb{N}$

Show that there exist a non-computable function $g : \mathbb{N} \to \mathbb{N}$

s.t. $g(n) = f(n) \quad \forall n < m$

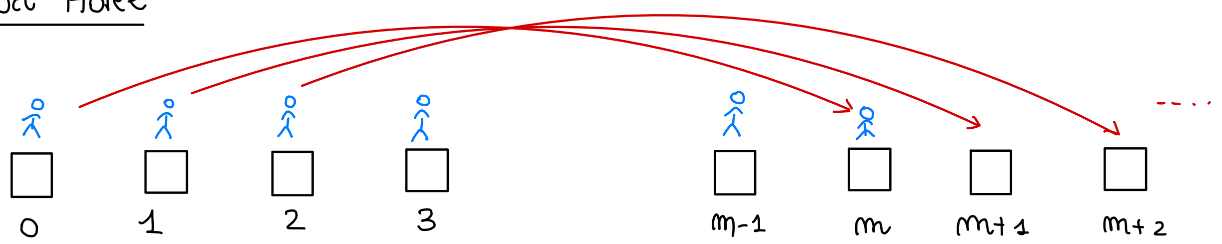
$$\begin{array}{c|ccc} & \varphi_0 & \varphi_1 & \varphi_2 \\ \hline 0 & \varphi_0(0) & \varphi_1(0) & \\ & \vdots & & \\ m-1 & \varphi_0(m-1) & \varphi_1(m-1) & \\ m & \bullet & \bullet & \\ m+1 & \bullet & \bullet & \\ m+2 & \bullet & \bullet & \bullet \end{array}$$

$$g(n) = \begin{cases} f(n) & n < m \\ \varphi_{n-m}(n) + 1 & n \geq m \text{ and } \varphi_{n-m}(n)\downarrow \\ 0 & n \geq m \text{ and } \varphi_{n-m}(n)\uparrow \end{cases}$$

then g is not computable since $\forall n$ $\quad g \neq \varphi_n$

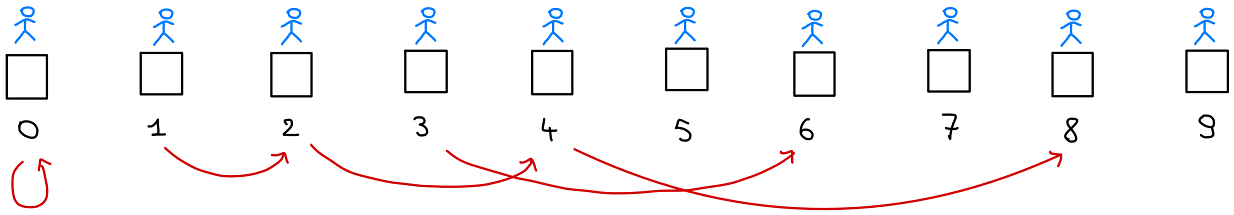
$$g(n+m) \neq \varphi_n(n+m)$$

<u>Hilbert' Hotel</u>



m persons coming

if infinitely (countably) many guests arrive?



$n \to 2n$

Alternative solution (for the exercise)

$$g(n) = \begin{cases} f(n) & n < m \\ \varphi_n(n) + 1 & n \geq m \text{ and } \varphi_n(n)\downarrow \\ 0 & n \geq m \quad '' \quad \varphi_n(n)\uparrow \end{cases}$$

g not computable

$$g \neq \varphi_n \quad \forall n \geq m$$

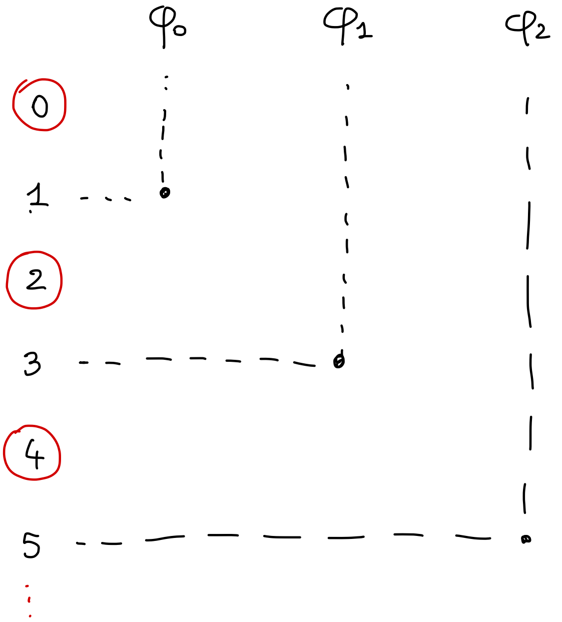
$$\varphi_0 \quad \varphi_1 \quad \ldots \quad \varphi_{m-1} \quad \varphi_m \quad \varphi_{m+1} \quad \text{-----}$$

recall: infinitely many programs (indices) for the same computable function for all computable functions h $\quad \exists\, n \geq m$ s.t. $\varphi_n = h$

hence $\quad g \neq \varphi_n = h \quad \Rightarrow \quad g$ not computable $\quad \square$

EXERCISE :

show that there is a function $g : \mathbb{N} \to \mathbb{N}$ total non-computable s.t. $\forall n$ if n is even then $f(n) = 0$



$$g(n) = \begin{cases} 0 & \text{if } n \text{ even} \\ \varphi_{\frac{n-1}{2}}(n) + 1 & \text{if } n \text{ odd } \& \ \varphi_{\frac{n-1}{2}}(n)\downarrow \\ 0 & \text{if } n \text{ odd } \& \ \varphi_{\frac{n-1}{2}}(n)\uparrow \end{cases}$$

$\to$ g is total

$\to$ $g(n) = 0$ if n is even

$\to$ g is not computable since $\forall n \ \ g \neq \varphi_n$

since $\varphi_n(2n+1) \neq g(2n+1)$

EXERCISE

Given $f_0, f_1, f_2, \dots \quad (f_i)_{i \in \mathbb{N}}$ functions

Define $f : \mathbb{N} \to \mathbb{N}$ s.t. $dom(f) \neq dom(f_i) \quad \forall i \in \mathbb{N}$.

PARAMETRISATION (SMN) THEOREM

Let $e \in \mathbb{N}$ and consider the function of two arguments computed by $P_e = \gamma^{-1}(e)$

$$f = \varphi_e^{(2)} : \mathbb{N}^2 \to \mathbb{N}$$

$$f(x,y) = \varphi_e^{(2)}(x,y)$$

Let $x \in \mathbb{N}$ be fixed

$$f_x : \mathbb{N} \to \mathbb{N}$$

$$f_x(y) = f(x,y)$$

f_x is computable

example : $f(x,y) = y^x$

$f_0(y) = y^0 = 1$

$f_1(y) = y^1 = y$

$f_2(y) = y^2$

$\vdots$

For all $x \in \mathbb{N}$ fixed f_x is computable i.e. there is $d \in \mathbb{N}$ s.t.

$$f_x = \varphi_d \qquad = \varphi_{s(e,x)}$$

d depends on e original program, x

hence there is a function

$$s : \mathbb{N}^2 \to \mathbb{N}$$

$$s(e,x) = d$$

The smn theorem says that s is <u>computable</u>

$f(x,y)$
```
def Pe(x,y)
   ...
   x
      y
   return result
```

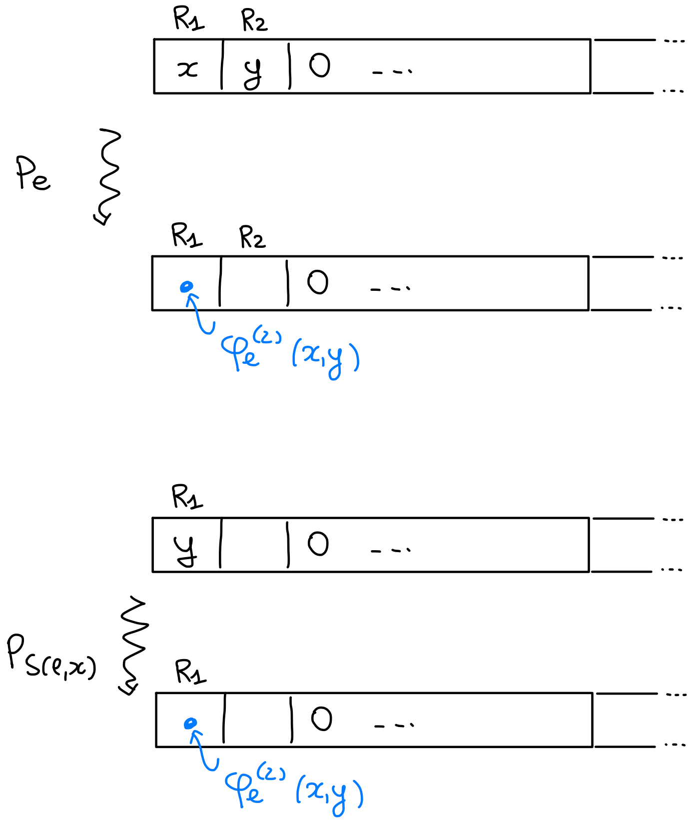
fix $x = 1$

$x = 2$

```
def Pe(~~x~~,y)
   ...
   ~~x~~ 1 2
      y
   return result
```

more ideas :

given a program P_e
(two arguments)



for each $x \in \mathbb{N}$ fixed
we want a program $P_{s(e,x)}$

we can "construct" $P_{s(e,x)}$

$$P_{s(e,x)} \begin{cases} \text{move } y \text{ to } R_2 \\ \text{write} \\ \text{execute } P_e = \gamma^{-1}(e) \end{cases}$$

The theorem says that

$s(e,x) = \gamma(\ \cdot\ )$ is computable.