Sobolev spaces (ch 5 Evans, PDE) / Brezis Functional analysis and PDEs

space of distributions provide useful tool to deal with generalized functions, nontheless it is sometimes better to work with Banach / Hilbert spaces (introduced by Sobolev in the '30)

$p \in [1, +\infty]$ $\quad U \subseteq \mathbb{R}^n$

$W^{1,p}(U) = \{ f \in L^p(U)$ such that $\forall \alpha \in \mathbb{N}^n$ $|\alpha| \leq 1$

$D^\alpha T_f = T_{v_\alpha} \in L^p(U)$ $\quad v_\alpha = D^\alpha f$ weak - α der.

all the distributional derivatives up to order 1 are weak derivatives and stay in L^p.

$k \in \mathbb{N}$

$W^{k,p}(U) = \{ f \in L^p(U)$ $\forall \alpha$ $|\alpha| \leq k$, $D^\alpha T_f = T_{v_\alpha}$

$v_\alpha \in L^p(U)$ $(v_\alpha = D^\alpha f)$ $\}$

$$\| \cdot \|_{W^{k,p}} = \sum_{|\alpha| \leq k} \| D^\alpha f \|_{L^p}$$

$(k = 0 \quad W^{0,p} = L^p !)$

by def. if $f \in W^{k,p}(U)$ $\int_U f\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_U D^\alpha f\, \phi\, dx \quad \forall \phi \in C_c^\infty(U)$

Theorem.

$(W^{k,p}(U), \|\cdot\|_{k,p})$ is a Banach space. For $p \in [1,+\infty)$ $W^{1,p}(U)$ is separable, for $p \in (1,+\infty)$ $W^{1,p}(U)$ is reflexive.

proof (only for $k=1$) $W^{1,p}(U), \|\cdot\|_{W^{1,p}}$

f_k Cauchy seq. in $W^{1,p}(U) \Rightarrow f_k$ is Cauchy in $L^p(U)$

$\frac{\partial}{\partial x_i} f_k$ is Cauchy in $L^p(U)$

$f_k \to f$ in $L^p(U) \Rightarrow T_{f_k} \to T_f \Rightarrow \frac{\partial}{\partial x_i} T_{f_k} \to \frac{\partial}{\partial x_i} T_f$

$\frac{\partial}{\partial x_i} f_k \to v_i$ in $L^p(U) \Rightarrow T_{\frac{\partial}{\partial x_i} f_k} \to T_{v_i}$

$v_i = \frac{\partial}{\partial x_i} f$ in weak sense.

$W^{1,p}(U) \subseteq L^p(U) \times L^p(U; \mathbb{R}^n)$ closed subspace

$f \longmapsto (f, (\frac{\partial}{\partial x_i} f)_i)$

for $p \in [1,+\infty)$ $W^{1,p}(U)$ is separable, for $p \in (1,+\infty)$ $W^{1,p}(U)$ is reflexive.

Obs for $p=2$ $\quad W^{k,2}(U) = H^k(U)$ is a Hilbert space

$$(f,g) = \sum_{|\alpha|\le k} \int_U D^\alpha f \cdot D^\alpha g \, dx$$

[also by Fourier transform $f \in L^2$ $\quad (1+|\xi|^2)^{k/2} \hat{f} \in L^2 \ldots$)
$H^s(U)$ $\quad s$ fractional.]

Elementary property: (1) $f, g \in W^{1,p}(U) \cap L^\infty(U) \Rightarrow f \cdot g \in W^{1,p}(U)$

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial}{\partial x_i} f \cdot g + f \frac{\partial}{\partial x_i} g.$$

(2) $F: \mathbb{R} \to \mathbb{R}$ $\quad F \in C^1(\mathbb{R})$ $\quad F(0)=0$ $\quad |F'| \le C$

$u \in W^{1,p}(U) \Rightarrow F(u) \in W^{1,p}(U)$

Def $\quad W_0^{k,p}(U) = \overline{C_c^\infty(U)}^{\|\cdot\|_{W^{k,p}}}$

$u \in W_0^{k,p}(U) \iff \exists\, \phi_n \in C_c^\infty(U)$ $\quad \phi_n \to u$ in $W^{k,p}$ norm.

$W_0^{k,p}(U) \subsetneq W^{k,p}(U)$ (apart from the case $U = \mathbb{R}^n$ as we will see).

$\boxed{\text{CLOSED SUBSPACE}}$

SOBOLEV SPACES in dim 1

Theorem Let $(a,b) \subseteq \mathbb{R}$ $\quad p \in [1,+\infty]$

$u \in W^{1,p}(a,b) \Longrightarrow \exists\, \bar{u} \in \mathcal{C}[a,b] \quad u = \bar{u}$ e.e.

u' its weak derivative

$$\bar{u}(x) - \bar{u}(y) = \int_y^x u'(t)\,dt \quad \forall\, x,y \in [a,b]$$

$u \in W^{1,p}(a,b) \Rightarrow u \in AC(a,b) = \{ u \in \mathcal{C}[a,b],\ u' \text{ exists a.e.},\ u' \in L^1(a,b)$

$\int_x^y u'(t)\,dt = u(y) - u(x) \}$

$\forall\, x,y \in [a,b]$

Actually $W^{1,1}(a,b) = AC(a,b)$ whereas $W^{1,p}(a,b) \subsetneq AC(a,b)$ $\quad p>1$

$W^{1,p}(a,b) \subseteq \mathcal{C}^{0,1-\frac{1}{p}}(a,b) \quad p>1$

__Proof__ fix $x_0 \in (a,b)$ $\quad \bar{u}(x) := \int_{x_0}^{x} u'(t)\,dt$ $\quad$ where u' is the weak derivative of u

1. $\bar{u} \in \mathcal{C}[a,b]$ – simple check.

$$|\bar u(x+h)-\bar u(x)| = \left|\int_{x_0}^{x+h} - \int_{x_0}^{x}\right| = \left|\int_{x}^{x+h} u'(t)\,dt\right|$$

if $u' \in L^1 \Rightarrow \frac{1}{h}\int_x^{x+h} u'(t)dt \to u'(x)$ a.e. $\Rightarrow \lim_{h\to 0^+} \int_x^{x+h} u'(t)\,dt = 0$
(Lebesgue)

if $u' \in L^p$ $p>1$ $\Rightarrow$ Holder $\left|\int_x^{x+h} u'(t)dt\right| \le \int_a^b |\chi_{(x,x+h)}(t)|\,|u'(t)|\,dt \le$

$$\le \|u'\|_{L^p}\; h^{1-\frac{1}{p}}.$$

so if $p>1$ $\bar u(x) \in C^{0,1-\frac{1}{p}}(a,b]$

(Lipschitz if $p=+\infty$
Holder if $p\in(1,+\infty)$ of
exp $1-\frac{1}{p}$.

② $\bar u' = u'$ in the weak sense. Fix $\phi \in C^\infty_c(a,b)$

$$\int_a^b \bar u \phi' \,dx = \int_a^b \int_{x_0}^x u'(t)\,\phi'(x)\,dx = \int_a^{x_0}\int_{x_0}^x u'(t)\phi'(x) + \int_{x_0}^b\int_{x_0}^x u'(t)\,\phi'(x) =$$

$$= -\int_a^{x_0}\int_x^{x_0} u'(t)\,\phi'(x) + \underbrace{\int_{x_0}^b \int_{x_0}^x u'(t)\,\phi'(x)}_{t} =$$

$a < x < x_0$ $\Rightarrow$ $a<t<x_0$
$x<t<x_0$ $\quad$ $a<x<t$

$x_0<x<b$ $\quad$ $t<x<b$
$x_0<t<x$ $\quad$ $x_0<t<b$

$$= -\int_a^{x_0}\int_a^t \phi'(x)\, u'(t) + \int_{x_0}^b \int_t^b \phi'(x) u'(t) = -\int_a^{x_0} \phi(t) u'(t) - \int_{x_0}^b \phi(t) u'(t)$$

$$\phi(a) = 0 = \phi(b)$$

$$= -\int_a^b u'(t)\,\phi(t).$$

Therefore $\bar{u}' = u'$ $\quad \int_a^b (\bar{u} - u)\,\phi'\,dt = 0$ for all $\phi \in C_c^\infty(a,b)$

(corollary of fundam. lemma calc var.) $\Rightarrow \bar{u} - u = c$ a.e.

$$\int_{x_0}^x u'(t) - u(x) = c = u(x_0) \qquad \square.$$

Note that in dim > 1 $\quad u \in W^{1,p}(U) \not\Rightarrow u \in C(U)$ in general.

$u(x) \in \frac{1}{|x|^r} \in L^p(B(0,1))$ $\quad rp < n$ $\quad r < \frac{n}{p}$.

$\phi \in C_c^\infty(B(0,1))$

$\frac{\partial u}{\partial x_i} = \frac{-r}{|x|^{r+2}} x_i$ if $x \neq 0$ $\quad \int_{B(0,1)} u \frac{\partial \phi}{\partial x_i} = \lim_{\varepsilon \to 0^+} \int_{B(0,1)\setminus B(0,\varepsilon)} u \frac{\partial \phi}{\partial x_i} =$ diverg $=$

it is the weak deriv.?

$$= \lim_{\varepsilon \to 0} \int_{B(0,1)\setminus B(0,\varepsilon)} \frac{\partial u}{\partial x_i}\cdot \phi + \int_{\partial B(0,\varepsilon)} u \cdot \phi\, \nu_i(x)\, dS$$

$\nu_i(x) = -\frac{x_i}{|x|}$

$$= \lim_{\varepsilon \to 0} \int_{B(0,1)\setminus B(0,\varepsilon)} \phi \frac{\partial u}{\partial x_i} + \underbrace{\int_{\partial B(0,\varepsilon)} \frac{1}{\varepsilon^{r}} \phi(x) \cdot \left(-\frac{x_i}{\varepsilon}\right)}_{\leq \frac{\|\phi\|_\infty}{\varepsilon^{r}} \cdot \varepsilon^{n-1} n \omega_n}$$

- if $0 < r < n-1$ $\quad -\frac{r x_i}{|x|^{r+2}}$ is the weak derivative
- $-\frac{r x_i}{|x|^{r+2}} \in L^p(B(0,1))$ $\quad \left| -\frac{r x_i}{|x|^{r+2}} \right| \sim \frac{r}{|x|^{r+1}} \in L^p(U)$ $\quad \boxed{(r+1)p < n}$

$p < n$

$$\frac{1}{|x|^r} \in W^{1,p}(B(0,1)) \iff \left[\begin{array}{l} p<n \\ r < \frac{n-p}{p} \end{array}\right]$$

(if $p \geq n$ $\quad \frac{1}{|x|^{r}} \notin W^{1,p}(B(0,1))$)

functions with irregularities may belong to $W^{1,p}$. $(p<n)$

Other examples:

$$u(x) = \log|x| \in W^{1,p}(B(0,1)) \quad \forall\, p < n$$

$\log|x| \in L^p(B(0,1))$ $\forall p \in [1, +\infty)$

$\frac{\partial}{\partial x_i} \log|x| = \frac{1}{|x|^2} x_i$ $\quad$ it is the weak derivative (check)

$\frac{1}{|x|^2} x_i \in L^p(B(0,1)) \iff p < n$

• $u(x) = \log \log \left(1+\frac{1}{|x|}\right) \in W^{1,n}(B(0,1))$ $\in W^{1,p}(B(0,1))\ \forall p \leq n$.

$x \neq 0$

$$u_{x_i} = \frac{1}{\log\left(1+\frac{1}{|x|}\right)} \left(\frac{|x|}{|x|+1}\right) \cdot \left(-\frac{x_i}{|x|^3}\right) = \frac{-x_i}{|x|^2(1+|x|)} \frac{1}{\log\left(1+\frac{1}{|x|}\right)}$$

I check it is the weak derivative.

$\phi \in C_c^\infty(B(0,1))$

$$\int_{B(0,1)} u \cdot \phi_{x_i} = \int_{B(0,1)\setminus B(0,\varepsilon)} u\, \phi_{x_i} \overset{\text{divergence}}{=} -\int_{B(0,1)\setminus B(0,\varepsilon)} u_{x_i}\, \phi + \underbrace{\int_{\partial B(0,\varepsilon)} u \cdot \phi \cdot \left(-\frac{x_i}{|x|}\right) dS}_{} = \leq \|\phi\|_\infty \underbrace{\log\left(\log\left(1+\frac{1}{\varepsilon}\right)\right)\varepsilon^{n-1}}_{\to 0}$$

$$\longrightarrow -\int_{B(0,1)} u_{x_i}\phi$$

$$|u_{x_i}| \sim \frac{1}{|x|(1+|x|)\log\left(1+\frac{1}{|x|}\right)} \in L^2(B(0,1)) \quad \text{since} \int_0^1 \frac{1}{(1+r)\log\left(1+\frac{1}{r}\right)} r^{n-2} < +\infty$$

In general $u \in W^{1,p}(U)$ $p \leq n$ is NOT CONTINUOUS.

whereas we will show $W^{1,p}(U) \subseteq C^{0,1-\frac{n}{p}}(U)$ $p > n$.