

METODI RUNGE-KUTTA

metodo a 2 punti:

$$p = f(y_n, t_n) \quad q = f(y_n + \alpha \Delta t p, t_n + \alpha \Delta t)$$

$$y_{n+2} = y_n + \Delta t (\beta_1 p + \beta_2 q)$$

$\varepsilon(\Delta t)$ errore da t_n a $t_n + \Delta t$ ho da
 $y(t_n) = y_n$

$$\varepsilon(\Delta t) = y(t_n + \Delta t) - y_{n+2}$$

↑ soluzione esatta.

$$q = f(y_n, t_n) + \frac{\partial f}{\partial y}(y_n, t_n) \alpha \Delta t p + \frac{\partial f}{\partial t}(y_n, t_n) \alpha \Delta t + O(\Delta t^2)$$

mentre ho da

$$y(t_n + \Delta t) = y(t_n) + \underbrace{\frac{dy}{dt}(t_n)}_{f(y_n, t_n)} \Delta t + \frac{1}{2} \underbrace{\frac{d^2 y}{dt^2}(t_n)}_{\frac{\partial f}{\partial y}(y_n, t_n) + \frac{\partial f}{\partial t}(y_n, t_n)} \Delta t^2 + O(\Delta t^3)$$

$$\varepsilon(\Delta t) = y(t_n) + f(t_n) \Delta t + \frac{1}{2} \frac{d^2 y}{dt^2}(t_n) \Delta t^2$$

$$- (y_n + \Delta t (\beta_1 q + \beta_2 q + \frac{d^2 y}{dt^2}(t_n) \alpha \beta_2 \Delta t) + O(\Delta t^3))$$

$$\varepsilon(\Delta t) = (1 - \beta_1 - \beta_2) f(t_n) \Delta t + (\frac{1}{2} - 2\beta_2) \frac{d^2 y}{dt^2}(t_n) \Delta t^2 + O(\Delta t^3)$$

Per una formula di ordine $O(\Delta t^2)$
 deve avere $1 - \beta_1 - \beta_2 = 0$

Per una formula di ordine $O(\Delta t^3)$
 anche $\frac{1}{2} - 2\beta_2 = 0$

Esempio: Eulero esplicito

$$\beta_2 = 0 \quad \alpha = 0 \quad \beta_1 = 1$$

Metodo Explicit mid-point Euler $O(\Delta t^2)$

$$\beta_1 = 0 \quad \beta_2 = 0 \quad \alpha = \frac{1}{2}$$

$$y_{n+1} = y_n + \Delta t f(y_n + \frac{1}{2} f(y_n, t_n) \Delta t, t_n + \frac{\Delta t}{2})$$

Metodo di Heun $O(\Delta t^3)$

$$\beta_1 = \beta_2 = \frac{1}{2} \quad \alpha = 1$$

$$y_{n+2} = y_n + \frac{1}{2} \Delta t (f(y_n, t_n) + f(y_n + f(y_n, t_n) \Delta t, t_n + \Delta t))$$

l'idea si estende a più punti:

RUNGE-KUTTA a 4 punti:

$$I_1 = y_n$$

$$I_2 = y_n + f(I_1, t_n) \Delta t / 2$$

$$I_3 = y_n + f(I_2, t_n + \Delta t / 2) \Delta t / 2$$

$$I_4 = y_n + f(I_3, t_n + \frac{\Delta t}{2}) \Delta t$$

$$y_{n+2} = y_n + (f(I_1, t_n) + 2f(I_2, t_n + \frac{\Delta t}{2}) + 2f(I_3, t_n + \frac{\Delta t}{2}) + f(I_4, t_n)) \frac{\Delta t}{6}$$