Theorem $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$

$(\forall T \in \mathcal{D}'(U)\ \exists\ \phi_n \in C_c^\infty(U)$ s.t. $\underbrace{T_{\phi_n} \to T}$ in the sense of distributions$)$

[NB $T_{\phi_n} \to T$ in the sense of distr $\int_U \phi_n \psi\, dx \to T(\psi)\ \forall \psi \in C_c^\infty(U)$

Lemma $\phi \in C_c^\infty(U)$

$$\eta(x) = \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 \end{cases} \qquad \int_{\mathbb{R}^n} \eta(x) = 1 \qquad \eta \in C_c^\infty(\mathbb{R}^n)$$

$$\eta_t(x) = \eta\left(\frac{x}{t}\right) t^{-n} \quad t>0 \qquad \eta(x) = \bar{\eta}(|x|) \qquad \operatorname{supp} \eta = B(0,1)$$

Let U' be an open neighborhood of $\operatorname{supp} \phi$.

$\operatorname{supp}(\phi * \eta_t) \subseteq U'$ for t small, $\phi * \eta_t \to \phi$ in $C_c^\infty(U')$

proof. 1) approximate T with distr. with compact support

$U = \bigcup_i V_i$ V_i open bdd, $V_{i+1} \supsetneq V_i \supsetneq \dots$ $\xi_i \in C_c^\infty(U)$ with

$0 \le \xi_i \le 1$ $\xi_i = 1$ on $\bar{V}_i$, $\operatorname{supp} \xi_i \subseteq V_{i+1}$.

$T_i = \xi_i T$ has compact support $\subseteq V_{i+1}$ $\left[\begin{array}{l} \operatorname{supp}\phi \subseteq U \setminus V_{i+1} \\ T_i(\phi) = T(\xi_i \phi) = 0 \end{array}\right]$

$T_i \to T$ in the sense of distribution.

$\left(\forall \phi \in \mathcal{C}_c^\infty(U) \quad \operatorname{supp}\phi \subseteq V_j \quad \exists j \quad T_k(\phi) = T(\xi_k \phi) = T(\phi) \quad \forall k \geq j \right.$

2) approximate T_i with $\phi_n \in \mathcal{C}_c^\infty(U)$.

$$T_i = \xi_i T \in \mathcal{D}'(\mathbb{R}^n) \quad \text{since} \quad \overset{\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)}{T_i(\phi)} = T(\xi_i \phi) \quad \xi_i \phi \in \mathcal{C}_c^\infty(U).$$

$\phi_t = T_i * \eta_t \in \mathcal{C}^\infty(\mathbb{R}^n)$ note that $\operatorname{supp} T_i \subseteq V_{i+1} \underset{\neq}{\subset} U$

$\forall \psi \in \mathcal{C}_c^\infty(U)$

$$\int_U (T_i * \eta_t)(x)\, \psi(x)\, dx = T_i(\tilde{\eta}_t * \psi) \overset{\text{since } \tilde{\eta}_t(x) = \eta_t(x)\,!}{=} T_i(\eta_t * \psi)$$

1) if $\operatorname{supp}\psi \subseteq U \setminus \overline{V_{i+1}} \implies T_i(\eta_t * \psi) = 0$ for $t \ll t_0$ by lemma

$\operatorname{supp}(T_i * \eta_t) \subseteq V_{i+1} \implies T_i * \eta_t \in \mathcal{C}_c^\infty(U)$

2) $\eta_t * \psi \to \psi$ in $\mathcal{C}_c^\infty(U)$ as $t \to 0$ $\quad \forall \psi \in \mathcal{C}_c^\infty(U)$

$\underline{T_i(\eta_t * \psi)} \to T_i(\psi) \implies T_i * \eta_t \to T_i$ in the sense of distr.

$\int T_i * \eta_t(x)\, \psi(x) = T_{(\phi_t)}(\psi)$

proved (for weak sol.) by H. Weyl in 1940

WEYL LEMMA: if $T \in \mathcal{D}'(\mathbb{R}^n)$ is a harmonic distribution $(-\Delta T = 0)$ then $T = T_u$ $u \in C^\infty(\mathbb{R}^n)$

(harmonic distributions are C^∞ functions).

Def: $\mathcal{F}$ linear diff. operator is HYPOELLIPTIC in $U \subseteq \mathbb{R}^n$ if $\forall T \in \mathcal{D}'(U)$ if $\mathcal{F}(T) \in C^\infty$ then T is C^∞.

Ex (LAPLACE OPERATOR) $\mathcal{F}u = \Delta u$ is hypoelliptic

HEAT OPERATOR $\mathcal{F}u = \frac{\partial}{\partial t}u - \Delta_x u$ (in $\mathbb{R}^{n+1} = \{(t,x),\ t \in \mathbb{R},\ x \in \mathbb{R}^n\}$)
is hypoelliptic

WAVE OPERATOR $\mathcal{F}u = \frac{\partial^2}{\partial t^2}u - \Delta_x u$ is NOT HYPOELLIPTIC.

Def $u \in L^1_{loc}(U)$ we say that u satisfies the

(SMV) for a.e. $x \in U$, $\forall r$ $B(x,r) \subseteq U$ $\quad u(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y)\, dS(y)$

(MV) for a.e. $x \in U$, $B(x,r) \subseteq U$ $\quad u(x) = \frac{1}{\omega_n r^n} \int_{B(x,r)} u(y)\, dy$.

Obs (SMV) $\iff$ (MV) ex.

$\Rightarrow$ $\frac{1}{\omega_n r^n} \int_{B(x,r)} u(y)\,dy = \frac{1}{\omega_n r^n} \int_0^r \int_{\partial B(x,s)} u(y)\,dS = \frac{1}{\omega_n r^n} \int_0^r u(x) \cdot n\omega_n s^{n-1} = u(x)$

$\Leftarrow$ $u(x) = \frac{1}{\omega_n r^n} \int_{B(x,r)} u(y)\,dy = \frac{1}{\omega_n r^n} \int_0^r \int_{\partial B(x,s)} u(y)\,dS = m(r)$

$m'(r) = \frac{-n}{\omega_n r^{n+1}} \int_{B(x,r)} u(y)\,dy + \frac{1}{\omega_n r^n} \int_{\partial B(x,r)} u(y)\,dS = \frac{-n}{r} u(x) + \frac{1}{\omega_n r^n} \int_{\partial B(x,r)} u(y)\,dS = 0$

Lemma 1 If u satisfies (SMV) (and (MV)) then $u \in C^\infty(U)$ and $\Delta u = 0$.

If $u \in C^2(U)$ and $\Delta u = 0 \Rightarrow u$ satisfies (MV) & (SMV).

Corollary: if $u \in L^1_{loc}(U)$ satisfies SMV $\Rightarrow u \in C^\infty$ and $\Delta u = 0$ in U

$\Rightarrow$: $u \in C^2$ and $\Delta u = 0 \Rightarrow u \in C^\infty$.

Proof $x \in U$ t such that $B(x,t) \subseteq U$.

$$u * \eta_t(x) = \int_{B(x,t)} u(y)\, \eta_t(x-y)\, dy = \int_0^t \int_{\partial B(x,s)} u(y)\, \eta_t(x-y)\, dS(y)\, ds =$$

$$= \eta_t(x) = \bar{\eta}_t(|x|) = \int_0^t \bar{\eta}_t(s) \int_{\partial B(x,s)} u(y)\, dS(y) = \text{SMV} = \int_0^t \bar{\eta}_t(s)\, n\omega_n s^{n-1}\, u(x) =$$

$$= u(x) \int_{B(x,t)} \eta_t(y)\,dy = u(x)$$

$\Big[\; u = u * \eta_t \quad \forall\, x \in U \ \operatorname{dist}(x, \partial U) > t$

$\Rightarrow u \in C^\infty(U)$.

$x \in U$ $\quad u \in C^2$ $\Delta u = 0 \Leftrightarrow u$ satisfies (SMV)

$$m(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y)\, dS(y) \quad \text{defined } \forall\, r < r_0 \quad B(x, r_0) \subset U.$$

$y = x + rz$

$$= \frac{1}{n\omega_n \cancel{r^{n-1}}} \int_{\partial B(0,1)} u(x+rz) \cdot \cancel{r^{n-1}}\, dS(z) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} u(x+rz)\, dS(z)$$

$$m'(r) = \frac{1}{n\omega_n} \int_{\partial B(0,1)} Du(x+rz)\cdot z \, dS(z) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} Du(y)\cdot \underbrace{\frac{(y-x)}{r}}_{\nu_{\partial B(x,r)}(y)} dS(y)$$

$$= \text{diverg.} = \boxed{\frac{1}{n\omega_n r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy}$$

If $\Delta u(y) = 0 \ \forall y \in U \implies m'(r) \equiv 0 \ \forall r \in (0, r_0) \to m(r) \equiv$ constant

$\implies m(r) = \lim_{s\to 0^+} m(s) = u(x) \quad \forall r \in (0,r_0) \implies u$ satisfies SMV

If u satisfies (SMV) $\implies m'(r) = 0 \ \forall r \in (0,r_0) \implies$ ⊛ $\int_{B(x,r)} \Delta u(y)\,dy = 0 \quad \forall r \in (0,r_0)$

$\implies \Delta u(x) = 0$ (by contradiction if $\Delta u(x) \neq 0$ say $\Delta u(x) > 0$

$\implies \exists r>0 \ \Delta u(y) > 0$ in $B(x,r)$ since $\Delta u \in C(U)$

$\implies \int_{B(x,r)} \Delta u(y) > 0$ in contradiction with ⊛

$\implies u_\varepsilon$ is C^∞ and $\Delta u_\varepsilon = 0 \implies u_\varepsilon$ satisfies (MV) and (SMV) $\implies$

$$u_\varepsilon * \eta_\delta = u_\varepsilon \quad \forall \delta \implies \underbrace{(T * \eta_\varepsilon) * \eta_\delta}_{\parallel} = T * \eta_\varepsilon \longrightarrow T \quad \text{as } \varepsilon \to 0$$

$$T(\eta_\varepsilon * \eta_\delta) = T(\eta_\delta * \eta_\varepsilon) = (T * \eta_\delta) * \eta_\varepsilon \to T * \eta_\delta$$

II.

proof of the Weyl lemma.

Let $T \in \mathcal{D}'(U)$ $\qquad \bar{t} > 0 \qquad U' = \{x \in U \quad d(x, \mathbb{R}^n \setminus U) > \bar{t}\}$

$T * \eta_t \in C^\infty(U')$ $\forall t \leq \bar{t}$ and $\Delta(T * \eta_t) = (\Delta T) * \eta_t = 0$.

so $T * \eta_t \in C^\infty(U')$ and $\Delta(T * \eta_t)(x) = 0 \quad \forall x \in U'$

since $\Delta(T * \eta_t) = 0 \ \forall x \in U' \Rightarrow$ $T * \eta_t$ satisfies (SMV) $\Rightarrow$

$\boxed{(T * \eta_t) * \eta_s(x) = (T * \eta_t)(x)}$ in $\{x \in U', \ d(x, \mathbb{R}^n \setminus U') > \bar{s}\} = U''$
$\forall s \leq \bar{s}$

$x \in U''$

$(T * \eta_t) * \eta_s(x) = \int_{U'} T * \eta_t(y)\, \eta_s(x-y)\, dy =$

prop. of convolutions ~~of~~ distr.

$= \int_{U'} \underbrace{T * \eta_t(y)} \cdot \underbrace{\eta_s^x(y)}\, dy \stackrel{\downarrow}{=} T(\tilde{\eta}_t * \eta_s^x) = T(\eta_t * \eta_s^x)$

$\eta_s^x(y) = \eta_s(x-y)$, $\qquad$ supp $\eta_s^x = B(x, s) \subseteq U' \quad \forall x \in U''$
$\eta_s^x \in C^\infty_c(U')$

$$\eta_t * \eta_s^x(y) = \int_{\mathbb{R}^n} \eta_t(z)\, \eta_s^x(y-z)\, dz = \int_{\mathbb{R}^n} \eta_t(z)\, \eta_s(x-y+z)\, dz$$

$$= \int_{\mathbb{R}^n} \eta_t(z)\, \eta_s(y-x-z)\, dz = \int_{\mathbb{R}^n} \eta_t(w-x)\, \eta_s(y-w)\, dw$$

$$\eta_s(x-y+z) = \eta_s(y-x-z)$$

$$w = x+z$$

$$= \int_{\mathbb{R}^n} \eta_t(x-w)\, \eta_s(y-w)\, dw = \int_{\mathbb{R}^n} \eta_t^x(w)\, \eta_s(y-w)\, dw =$$

$$= \eta_t^x * \eta_s(y) \quad (*)$$

$$(T * \eta_t) * \eta_s(x) \underset{\circledast}{=} T(\eta_t * \eta_s^x) \overset{\text{by } (*)}{=} T(\eta_s * \eta_t^x) \overset{\text{by } \circledast}{=} (T * \eta_s) * \eta_t(x)$$

$\Longrightarrow$ In conclusion we have

$$T * \eta_t(x) = (T * \eta_s) * \eta_t(x) \qquad \forall x \in U''$$

send $t \to 0^+$

$$(T * \eta_s) * \eta_t \longrightarrow T * \eta_s \quad \text{in } U''$$ in C^∞ and also in the sense of distribution

since $$\int_{U''} (T * \eta_s) * \eta_t \, \varphi(y)\,dy \longrightarrow \int_{U''} (T * \eta_s)\, \varphi(y)\,dy \quad \forall \varphi \in C_c^\infty(U'')$$

$$T * \eta_t(x) \longrightarrow T \quad \text{in the sense of distribution}$$

$$\Longrightarrow \quad T * \eta_t = T \quad \text{as distributions}$$

(This means that T is the distribution associated to the smooth function $T * \eta_t$.