Borel measures on $\mathbb{R}$ / on $\mathbb{R}^n$

X set, $\Sigma \subseteq \mathcal{P}(X) = \{ A \subseteq X \text{ subsets} \}$

Σ is a σ-algebra if $\phi \in \Sigma$, it is closed by passage to the complement if $A \in \Sigma \Rightarrow X \setminus A \in \Sigma$;
it is closed by countable union

$A_i \in \Sigma \quad i \in \mathbb{N} \implies \bigcup_i A_i \in \Sigma.$

IN GENERAL
if $A \in \Sigma$ and $B \subsetneq A$
it is not true
that $B \in \Sigma$.

smallest σ-algebra $\{\phi, X\}$
largest σ-algebra $\mathcal{P}(X)$

$C \subseteq \mathcal{P}(X)$ $\quad C$ a set of subsets of X

$\Sigma_C = \sigma$-algebra generated by C = smallest σ-algebra which contains all elements in C.

$X = \mathbb{R}$ $\quad C = \{ \underbrace{(a,b)}, \quad a < b \quad a, b \in \mathbb{R} \}$

$$(a,b) = \{ x \in \mathbb{R} \quad a < x < b \}$$

$\Sigma_C =$ σ-algebra generated by $C = \mathcal{B}(\mathbb{R}) = \underline{\text{Borel}}$
$\underline{\text{σ-algebra}}$

$\mathcal{B}(\mathbb{R})$ contains (a,b) $\quad (a,+\infty) \quad (-\infty,b)$

$(-\infty, b]$, $[a, +\infty)$ $\quad \displaystyle \overset{\shortparallel}{\bigcup_{n=1}^{+\infty} (a, b+n)} \quad \overset{\shortparallel}{\bigcup_{n=1}^{\infty} (a-n, b)}$

$[a,b] = \mathbb{R} \setminus [\underbrace{(-\infty, a) \cup (b, +\infty)}]$

$[a,b)$, $(a,b]$ ...

$\mathcal{B}(\mathbb{R}) = \sigma$-algebra generated by $\{[a,b] , a<b\}$

by $\{(a,b]\ a<b\}$

by $\{[a,b)\ a<b\}$

If X is a set where there is a DISTANCE

$$\boxed{\begin{aligned} d : X \times X &\to [0,+\infty) \\ x,y &\mapsto d(x,y) \end{aligned}}$$

$d(x,y)$ = distance between x,y

1) $d(x,x)=0$, $d(x,y) \geq 0$ and $d(x,y)=0 \iff x=y$

2) $d(x,y) = d(y,x)$

3) $d(x,y) \leq d(x,z) + d(z,y)$

d is a distance if 1) 2) 3) hold.

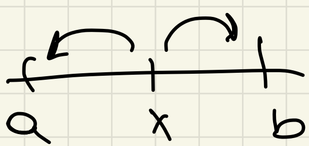
$\mathcal{B}(X)$ = σ-algebra generated by the family $C = \{ I_r(x) \subset X, \text{ for } x \in X \quad r > 0 \}$

$$I_r(x) = \{ y \in X \quad d(y,x) < r \}$$

If $X = \mathbb{R}$ $\quad d(x,y) = |x-y|$

$$I_r(x) = (x-r, x+r) = \{ y \in \mathbb{R} \quad |x-y| < r \}$$

$(a,b) = (x-r, x+r)$ $\qquad x = \frac{a+b}{2} \quad r = \frac{b-a}{2}$



$\mathcal{B}(\mathbb{R}^n)$ $\qquad n>1 \qquad \mathbb{R}^n \qquad d(x,y) = |x-y| =$

$$= \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2}$$

$\downarrow$

σ- algebra in $\mathbb{R}^n$ generated by

$$I_r(x) = \{ y \in \mathbb{R}^n \quad \sqrt{(x_1-y_1)^2 + \dots + (x_n-y_n)^2} < r \}$$

$$(x_1-y_1)^2 + \dots + (x_n-y_n)^2 < r^2$$

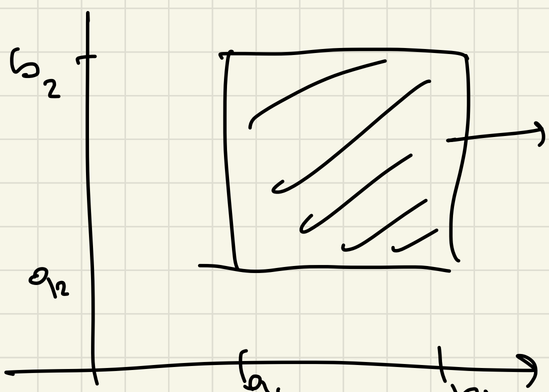
$= B(x,r) =$ open ball in $\mathbb{R}^n$ centered at x and with radius r.

$$\mathcal{B}(\mathbb{R}^n) = \underbrace{\mathcal{B}(\mathbb{R}) \times \dots \times \mathcal{B}(\mathbb{R})}_{n \text{ times}}$$

$\mathcal{B}(\mathbb{R}^n)$ is the smallest σ-algebra generated by $B(x,r)$ $x \in \mathbb{R}^n$, $r > 0$

it is also the smallest σ-algebra generated by rectangulars

$(a_1,b_1) \times (a_2,b_2) \times \dots \times (a_n,b_n)$



$\to (a_1;b_1) \times (a_2,b_2)$

$\mathbb{C}$ = complex number $\cong \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$\mathcal{B}(\mathbb{C})$ ...

I fix
$\forall$ X a set , Σ a σ-algebra on X.

$$\mu : \Sigma \longrightarrow [0, +\infty]$$
$$A \longmapsto \mu(A) \geq 0 \quad \text{(possibly } \mu(A) = +\infty)$$

is called a measure

if $\mu(\emptyset) = 0$

and μ is $\underline{\underline{\sigma}}$-addictive ($\sigma$ = countable)

that is if $A_i \in \Sigma$ $\quad A_i \cap A_j = \emptyset$, $i \neq j$ $\quad \mu\left(\bigcup_{i=0}^{+\infty} A_i\right) = \sum_{i=0}^{+\infty} \mu(A_i)$

If μ is a measure defined on Σ, σ-algebra of X

we say that (X, Σ, μ) is a measure space

set $\quad$ σ-algebra $\quad$ measure

Properties of measures:

1) monotonicity with respect to inclusion.

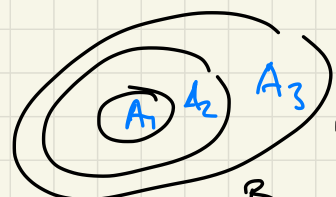
$$A \subseteq B \quad A, B \in \Sigma \implies \mu(A) \le \mu(B)$$

$$B = A \cup (B \setminus A) \qquad \mu(B) = \mu(A) + \mu(B \setminus A)$$

2) continuity

family $A_i \in \Sigma \qquad A_i \subseteq A_{i+1} \quad \forall i$

$$\boxed{\mu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \lim_{n \to +\infty} \mu(A_n)}$$



$B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus A_2$

...

$A_i \supseteq A_{i+1} \quad \forall i$

$\mu(A_{i_0}) \neq +\infty$

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to +\infty} \mu(A_n)$$

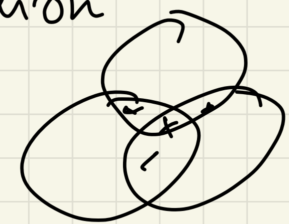
$B_1 = A_1 \setminus A_2$

$B_2 = A_2 \setminus A_3$

3) σ-subadditivity

$A_i \in \Sigma$ ~~A~~ possibly with intersection

$$\underline{\mu\left(\bigcup_{i=1}^{\infty} A_i\right)} \leq \underline{\sum_{i=1}^{\infty} \mu(A_i)}$$



X a space with a distance $\mathcal{B}(X)$

I say that μ is a Borel measure if it's defined on $\mathcal{B}(X)$.

(in $\mathbb{R}^n$ Borel measures will be defined on $\mathcal{B}(\mathbb{R}^n)$)

Let's fix (X, Σ, μ) a metric space $\mu : \Sigma \to [0, +\infty]$ a measure.

$\overline{\Sigma}$ = completion of Σ with respect to μ

$\overset{!}{=} \Sigma \cup \{A \subseteq X \setminus \Sigma$, such that $\exists B \in \Sigma$, $\mu(B) = 0$ and $A \subseteq B\}$.

I will extend μ putting $\mu(A) = 0$

$\mu : \overline{\Sigma} \to [0, +\infty]$

(passage to the completion is an abstract procedure, I'm adding "negligible sets", subsets of sets of measure zero

if $A \in \Sigma$ $\mu(A) = 0$ from the point of view of the measure, the information we have on A are the same than the information we have on $\emptyset$.

we say that a property on elements of (X, Σ, μ) holds ALMOST EVERYWHERE if it holds $\forall x \in X \setminus A$ where $\mu(A) = 0$.

Def (X, Σ, μ) we say that $\boxed{\mu \text{ is finite} \quad \mu(X) < +\infty}$

$(X \in \Sigma \ \forall \ \sigma\text{-algebra})$

μ is σ-finite if $X = \underbrace{\bigcup_{n=1}^{\infty} A_n}$ $\quad A_n \in \Sigma$

$\mu(X) = +\infty$

such that $\mu(A_n) < +\infty$.

Prop $F : \mathbb{R} \longrightarrow \mathbb{R}$, F monotone non-decreasing

$x < y \qquad x, y \in \mathbb{R}$

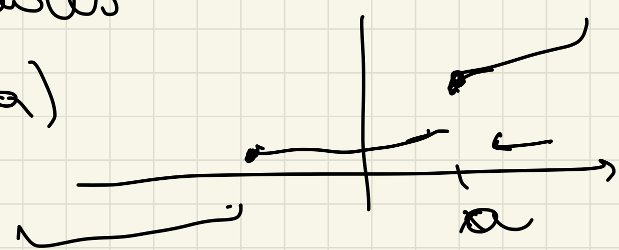
$F(x) \leq F(y)$

(increasing but not necessarily strictly increas.)

F right continuous

$$\lim_{x \to a^+} F(x) = F(a)$$

$x \to a \quad x > a$



I define

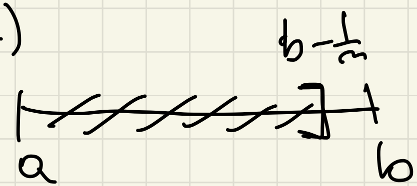
$$\mu_F(a,b] := F(b) - F(a).$$

Proposition μ_F can be extended to a Borel measure. (a measure on all the Borel sets).

$$\mu_F \underbrace{(a,b)} = \mu_F\left[\bigcup_{n=1}^{+\infty} \left(a, b-\frac{1}{n}\right]\right] = \lim_{n \to +\infty} \mu_F\left(a, b-\frac{1}{n}\right]$$

$$= \lim_{n} F\left(b-\frac{1}{n}\right) - F(a)$$

$$= \lim_{x \to b^-} F(x) - F(a)$$



$\forall n \in \mathbb{N}$

$$b - \frac{1}{n} \in \bigcup_{k=1}^{\infty} \left(a, b-\frac{1}{k}\right]$$

$$b \notin \bigcup_{k=1}^{\infty} \left(a, b-\frac{1}{k}\right]$$

Obs

$$\mu_F(a,b] - \mu_F(a,b) = \mu_F\left[(a,b] \setminus (a,b)\right] =$$

$$= \mu_F(\{b\}) = F(b) - \lim_{x\to b^-} F(x) \geq 0$$

If F is continuous in b $\left(\lim_{x\to b} F(x) = F(b)\right)$

$$\Longrightarrow \mu_F\{b\} = 0$$

If F has a jump in b $\mu_F\{b\} =$ height of the jump.

$$\mu_F(\mathbb{R}) = \mu_F\left[\bigcup_{n=1}^{+\infty} (-n,n]\right] =$$

$$= \lim_{n\to+\infty} \mu_F(-n,n] = \lim_{x\to+\infty} F(x) - F(-x) =$$

$$= \lim_{x \to +\infty} F(x) - \lim_{y \to -\infty} F(y)$$

μ_F is finite $\Longleftrightarrow$ $\underline{\lim_{x \to +\infty} F(x) < +\infty}$

$\underline{\lim_{x \to -\infty} F(x) > -\infty}$

(it is finite if F has finite limits at $+\infty, -\infty$
it is σ-finite in all the other cases)

$\boxed{\mu_F \text{ is always } \sigma\text{-finite or finite}}$

$$\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n]$$

$$\mu_F(-n, n] = F(n) - F(-n) \geq 0$$

$$\in \mathbb{R}$$

Theorem

Every μ finite or σ-finite Borel measure on $\mathbb{R}$ is actually μ_F for some $F : \mathbb{R} \to \mathbb{R}$ right continuous and monotone non-decreasing.

F is unique up to addition of constants

$\forall c \in \mathbb{R}$

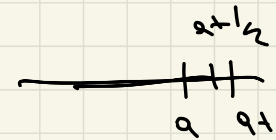
$$\underbrace{\mu_{F(x)+c}}_{\downarrow} = \mu_{F(x)}$$

$$\mu_{F(x)+c}(a,b] = F(b)+\cancel{c} - (F(a)+\cancel{c})$$

F is called the "cumulative" distribution function

μ a σ-finite / finite Borel measure

$$\boxed{F(x) = \begin{cases} \mu(0,x] & x>0 \\ 0 & x=0 \\ -\mu(x,0] & x<0 \end{cases}}$$

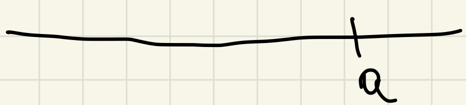


monotonicity of μ w.r.t. inclusion $\Longrightarrow$ F non decreasing

continuity of μ $\Longrightarrow$ right continuity of F

$a>0$

$$\lim_{x\to a^+} F(x) = \lim_{n\to+\infty} F\left(a+\frac{1}{n}\right) = \lim_{n\to+\infty} \mu\left(0, a+\frac{1}{n}\right]$$



$$= \mu\left(\bigcap_{n=1}^{+\infty} \left(0, a+\frac{1}{n}\right]\right) = \mu_F(0,a] = F(a).$$

$\mu(-\infty,+\infty] < +\infty$

If μ is finite we may also define

$$\boxed{G(x) := \mu(-\infty, x]} < +\infty \quad \begin{cases} \text{non decreasing} \\ \text{right cont.} \end{cases}$$

$x > 0$

$$F(x) + \mu(-\infty,0] = \mu(0,x] + \mu(-\infty,0] = G(x)$$

$x = 0$

$$F(0) + \mu(-\infty,0] = 0 + \mu(-\infty,0] = G(0)$$

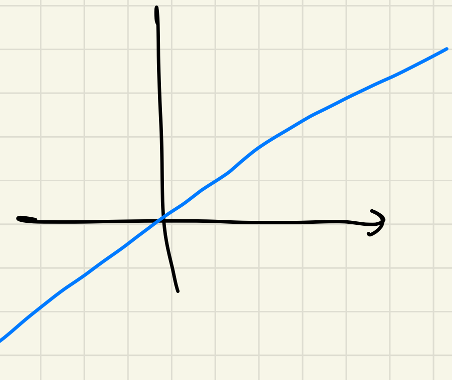
$$(-\infty,0] = (-\infty,x] \cup (x,0]$$

$x < 0$

$$F(x) + \mu(-\infty,0] = \underbrace{-\mu(x,0] + \mu(-\infty,0]} = \mu(-\infty,x] = G(x)$$

$$F + \overbrace{\mu(-\infty,0]}^{c} = G \qquad \mu_F = \mu_G$$

Ex. $F(x) = x$



μ_F is called the Lebesgue measure $\mathcal{L}$

(it is a measure which associates to intervals their length

$\mu_F(a,b) = \mu_F(a,b] = b-a \qquad (b>a)$

$\mu_F\{a\} = 0$ because F is continuous

$\mathcal{L}$ is σ-finite (NOT finite)

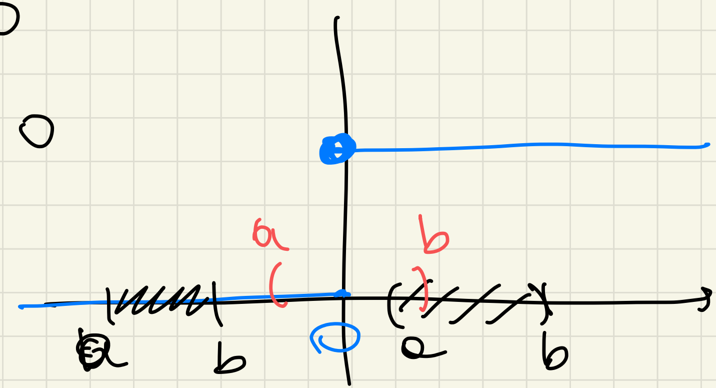
$\overline{\mathcal{B}}(\mathbb{R})$ completion of $\mathcal{B}(\mathbb{R})$ with respect to $\mathcal{L}$

$= \mathcal{M}$ Lebesgue measurable sets

(it is NOT all the $\mathcal{P}(\mathbb{R})$!)

there are subsets of $\mathbb{R}$ which are NOT MEAS by Lebesgue (it is not possible to define a notion of length which is coherent and applicable to all possible subsets of $\mathbb{R}$).

$$F(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



$\mu_F = \delta_0$ (Dirac measure)

$$\delta_0(a,b] = F(b) - F(a)$$

$$\to 0 \quad \begin{cases} \text{if } a < b < 0 \\ \text{if } 0 \leq a < b \end{cases}$$

$$\to 1 \quad a < 0 \leq b$$

$\boxed{\delta_0\{0\} = 1}$ $\quad \delta_0\{a\} = 0 \ \ \forall a \neq 0$

$A \subseteq \mathcal{B}(\mathbb{R})$

$$\delta_0(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$$

$$\overline{\mathcal{B}(\mathbb{R})} = \mathcal{P}(\mathbb{R})$$

$\mathbb{R} \setminus \{0\}$ is a set of measure 0

$\delta_0(\mathbb{R} \setminus \{0\}) = 0$

$\overline{\mathcal{B}}(\mathbb{R})$ contains all subsets of $\mathbb{R} \setminus \{0\}$

it contains also $\{0\}$

$\delta_0 : \mathcal{P}(\mathbb{R}) \longrightarrow [0, +\infty]$

$c \in \mathbb{R}$ $\qquad \delta_c = \mu_{F_c}$ $\qquad F(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$

Dirac measure centered at c.

$\delta_c(A) = \begin{cases} 1 & \text{if } c \in A \\ 0 & \text{if } c \notin A \end{cases}$.

$F(x) = [x] =$ integer part of x

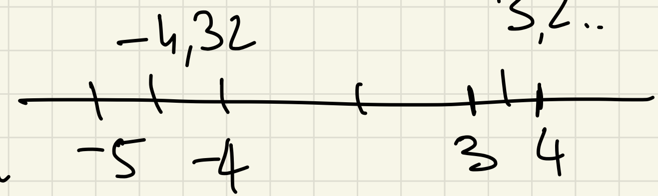
right cont and increasing

$[x] \in \mathbb{Z}$

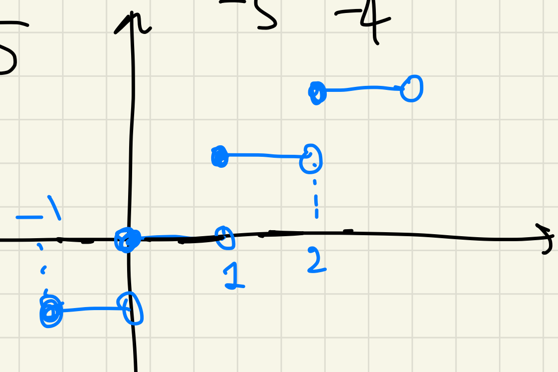
$[x] = \{$ biggest $z \in \mathbb{Z}$ such that $z \le x$. $\}$

$[3,271] = 3$

$[-4,32] = -5$

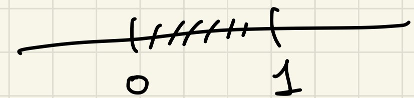


$[-0,3] = -1$



μ_F "counting measure"

$$\mu_F(A) = \# \{ z \in \mathbb{Z},\ z \in A \}$$



number

$A \subseteq \mathcal{B}(\mathbb{R})$ $\quad \mu_F(A) =$ number of integers which are in A

$k \in \mathbb{Z}$

$$\underbrace{\mu_F\{k\} = 1} \qquad \underbrace{\mu_F\{x\} = 0} \quad x \notin \mathbb{Z}.$$

$$\overline{\mathcal{B}(\mathbb{R})} = \mathcal{P}(\mathbb{R}) \qquad \cdot\ \mu_F \underbrace{(n, n+1)} = 0 \quad \forall n \in \mathbb{Z}$$

ex. Ω set

$\mathcal{F}$ = σ-algebra on Ω

$\mathbb{P} : \mathcal{F} \to [0,1]$ is a probability measure

(if it is a measure only taking values in $[0,1]$

$\mathbb{P}(\Omega) = 1$ ...