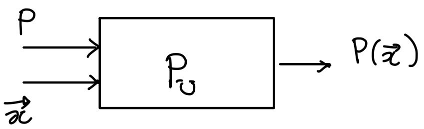
COMPUTABILITY (05/11/2024)



* Enumeration of URM programs

set X countable if $|X| \leq |\mathbb{N}|$

which there is $f : \mathbb{N} \to X$ (total) surjective (enumeration of X)

$$\underbrace{f(0) \quad f(1) \quad f(2) \quad f(3) \quad f(4) \quad \cdots\cdots}_{X}$$

if f is also injective $\leadsto$ bijective enumeration

f "effective"

Lemma: there are bijective enumerations of (effective)

1. $\mathbb{N}^2$
2. $\mathbb{N}^3$
3. $\bigcup_{k \geq 1} \mathbb{N}^k$

① $\pi : \mathbb{N}^2 \to \mathbb{N}$

$$\pi(x,y) = \underbrace{\overbrace{2^x (2y+1)}^{n+1} - 1}_{n} \qquad \text{computable}$$

$\pi^{-1} : \mathbb{N} \to \mathbb{N}^2$ $\leftarrow$ effective

$\pi^{-1}(n) = (\pi_1(n), \pi_2(n))$

$\pi_1 : \mathbb{N} \to \mathbb{N}$

$$\pi_1(n) = (n+1)_1 \qquad \longleftarrow \qquad \text{computable}$$

$\pi_2 : \mathbb{N} \to \mathbb{N}$

$$\pi_2(n) = \left(\frac{n+1}{2^{\pi_1(n)}} \dot{-} 1\right) \Big/ 2 = qt(2, qt(2^{\pi_1(n)}, n+1) \dot{-} 1)$$

② $\nu : \mathbb{N}^3 \to \mathbb{N}$

$\nu(x,y,z) = \pi(x, \pi(y,z))$ computable

$\nu^{-1} : \mathbb{N} \to \mathbb{N}^3$

$\nu^{-1}(m) = (\nu_1(m), \nu_2(m), \nu_3(m))$

$\nu_1, \nu_2, \nu_3 : \mathbb{N} \to \mathbb{N}$ computable

$\nu_1(m) = \pi_1(m)$

$\nu_2(m) = \pi_1(\pi_2(m))$

$\nu_3(m) = \pi_2(\pi_2(m))$

③ $\tau : \bigcup_{k \geq 1} \mathbb{N}^k \to \mathbb{N}$

$$\tau(x_1, \dots, x_k) = \prod_{i=1}^{k} p_i^{x_i} \dot{-} 1$$

not injective!

$$\tau(1,2,0) = p_1^1 \cdot p_2^2 \cdot \overset{1}{p_3^0} \dot{-} 1$$

$$\tau(1,2) = p_1^1 \cdot p_2^2 \dot{-} 1$$

variation

$$\tau(x_1, \dots, x_k) = \left(\prod_{i=1}^{k-1} p_i^{x_i}\right) \cdot p_k^{x_k+1} \dot{-} 2$$

$$\tau^{-1} : \mathbb{N} \to \bigcup_{k \geq 1} \mathbb{N}^k$$

$$\tau^{-1}(m) = \underbrace{(a(m,1) \quad a(m,2) \quad \dots \quad a(m, \ell(m)))}_{\ell : \mathbb{N} \to \mathbb{N} \qquad \ell(m) = k}$$

$a : \mathbb{N}^2 \to \mathbb{N}$

$$m = \tau(\text{------}) = \left(\prod_{i=1}^{k-1} p_i^{x_i}\right) \cdot p_k^{x_k+1} \dot{-} 2$$

$$m+2 = \left(\prod_{i=1}^{k-1} p_i^{x_i}\right) \cdot p_k^{x_k+1}$$

$\ell(m) =$ largest k s.t. p_k divides $m+2$ [computable, exercise]

$$a(m,i) = \begin{cases} (m+2)_i & 1 \leq i \leq \ell(m)-1 \\ (m+2)_i \dot{-} 1 & i = \ell(m) \end{cases}$$

[computable]

<u>OBSERVATION</u> : Let $\mathcal{P}$ be the set of URM programs

There is a bijective (effective) enumeration $\gamma : \mathcal{P} \to \mathbb{N}$

Let $\mathcal{I} = \{ Z(n),\ S(n),\ T(m,n),\ J(m,n,t) \mid m,n,t \in \mathbb{N} \}$

we consider

$$\beta : \mathcal{I} \to \mathbb{N}$$

0	1	2	3	4	5	6	7	8	9	10	... $\mathbb{N}$
Z(1)	S(1)	T( )	J( )	Z(2)	S(2)	T( )	J( )	Z(3)	S(3)	T( )	J( )

$$\beta(Z(n)) = 4 \cdot (n \dot{-} 1)$$

$$\beta(S(n)) = 4 \cdot (n \dot{-} 1) + 1$$

$$\beta(T(m,n)) = 4 \cdot \pi(m \dot{-} 1, n \dot{-} 1) + 2$$

$$\beta(J(m,n,t)) = 4 * \nu(m \dot{-} 1, n \dot{-} 1, t \dot{-} 1) + 3$$

$\beta^{-1} : \mathbb{N} \to \mathcal{I}$ $\qquad x \leadsto \quad r = rm(4, x)$, $\quad q = qt(4, x)$

$$\beta^{-1}(x) = \begin{cases} Z(q+1) & \text{if } r = 0 \\ S(q+1) & \text{if } r = 1 \\ T(\pi_1(q)+1, \pi_2(q)+1) & \text{if } r = 2 \\ J(\nu_1(q)+1, \nu_2(q)+1, \nu_3(q)+1) & \text{if } r = 3 \end{cases}$$

Given a program $P \in \mathcal{P}$

$$P = \begin{cases} I_1 \\ I_2 \\ \vdots \\ I_s \end{cases} \qquad \gamma(P) = \tau(\beta(I_1)\ \beta(I_2)\ \dots\ \beta(I_s))$$

inverse

$$\gamma^{-1} : \mathbb{N} \to \mathcal{P}$$

$$\gamma^{-1}(n) = P \begin{cases} I_1 \\ \vdots \\ I_{\ell(n)} \end{cases} \qquad I_i = \beta^{-1}(a(n,i)) \qquad 1 \le i \le \ell(n)$$

$*$ γ fixed enumeration (bijective) of URM programs

$\gamma(P)$ (Gödel) number of P

given $n \in \mathbb{N}$ $P_n = \gamma^{-1}(n)$

<u>Example</u>:

$*$

$$P \begin{cases} T(1,2) & \overset{\beta}{\rightsquigarrow} & 4 * \pi(1-1, 2-1) + 2 = 4 * \overset{\substack{2^0(2\cdot 1+1)-1 = 2 \\ \uparrow}}{\pi(0,1)} + 2 = 10 \\ S(2) & \rightsquigarrow & 4 * (2-1) + 1 = 5 \\ T(2,1) & \rightsquigarrow & = 6 \end{cases}$$

$$\gamma(P) = \tau(10 \quad 5 \quad 6) =$$

$$= p_1^{10} \cdot p_2^{5} \cdot p_3^{6+1} \dot{-} 2 = 2^{10} \cdot 3^5 \cdot 5^7 \dot{-} 2$$

$$= 19\,439\,999\,998$$

$*$ P' $S(1)$

$$\gamma(P') = \tau(\beta(S(1))) = \tau(4 * (1-1) + 1) = \tau(1)$$

$$= p_1^{1+1} \dot{-} 2 = 2^2 \dot{-} 2 = 2$$

$*$ given $n = 40$

what is $P_{40} = \gamma^{-1}(40)$?

$$\underbrace{\left(\prod_{i=1}^{k-1} p_i^{x_i}\right) \cdot p_k^{x_k+1} \dot{-} 2}_{n}$$

$$n+2 = 42 = \underset{\substack{\uparrow \\ p_1}}{2^1} \cdot \underset{\substack{\uparrow \\ p_2}}{3^1} \cdot \underset{\substack{\uparrow \\ p_3}}{5^0} \; \underset{\substack{\uparrow \\ p_4}}{7^{0+1}}$$

$\ell(40) = 4$

$$\gamma^{-1}(40) \begin{cases} I_1 & \beta^{-1}(1) & S(1) \\ I_2 & \beta^{-1}(1) & S(1) \\ I_3 & \beta^{-1}(0) & Z(1) \\ I_4 & \beta^{-1}(0) & Z(1) \end{cases}$$

* Fixed $\gamma : \mathcal{P} \to \mathbb{N}$

this induces an enumeration of the computable function

$$\varphi_n^{(k)} : \mathbb{N}^k \to \mathbb{N}$$

function of k arguments computed by $P_n = \gamma^{-1}(n)$ (i.e. $f_{P_n}^{(k)}$)

$$W_n^{(k)} = \{ \vec{x} \in \mathbb{N}^k \mid \varphi_n^{(k)}(\vec{x}) \downarrow \} = dom(\varphi_n^{(k)}) \subseteq \mathbb{N}^k$$

$$E_n^{(k)} = \{ \varphi_n^{(k)}(\vec{x}) \mid \vec{x} \in W_n^{(k)} \} \subseteq \mathbb{N}$$

When $k=1$ we omit it

φ_n for $\varphi_n^{(1)}$

Example :

$$\varphi_{40} : \mathbb{N} \to \mathbb{N}$$

$$\varphi_{40}(x) = 0 \quad \forall x$$

$$W_{40} = \mathbb{N} \qquad E_{40} = \{0\}$$

successor $\qquad$ successor

$$\underbrace{\varphi_0 \quad \varphi_1 \quad \varphi_2 \quad \varphi_3 \quad \cdots\cdots \quad \varphi_{19\,433\,999\,998}}$$

enumeration of all unary computable functions

with repetitions (infinitely many)

$$|\mathcal{C}^{(1)}| \leq |\mathbb{N}|$$

$$|\mathcal{C}^{(k)}| \leq |\mathbb{N}| \quad \forall k$$

$$\mathcal{C} = \bigcup_{k \geq 1} \mathcal{C}^{(k)} \quad \text{denumerable}$$

$$|\mathcal{C}| \leq |\mathbb{N}|$$

<u>Exercise</u>: $\mathcal{R}$ partial recursive function

least class of functions which

→ contains the basic function

→ closed under
→ composition
→ primitive recursion
→ minimalisation

Originally introduced by Gödel-Kleene $\mathcal{R}_0$

least class of functions which

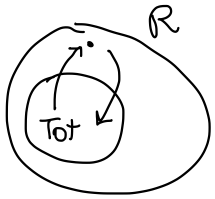
→ contains the basic function

→ closed under
→ composition
→ primitive recursion
→ minimalisation only when producing a total functions

$$\mathcal{R}_0 \subseteq \mathcal{R} \cap Tot$$

$? \supseteq$

not obvious since I can obtain total functions by minimalisation of partial functions in $\mathcal{R}$



$f : \mathbb{N}^2 \to \mathbb{N}$

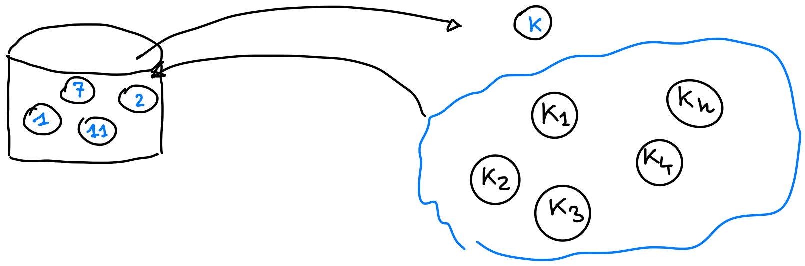
$$f(x,y) = \begin{cases} 1 & \text{if } y < x \\ 0 & \text{if } y = x \\ \uparrow & \text{if } y > x \end{cases}$$

$$= \underbrace{sg(x \dot{-} y)}_{\substack{1 \quad y<x \\ 0 \quad y \geq x}} + \underbrace{\mu z. (y \dot{-} x)}_{\substack{0 \quad \text{if } y \leq x \\ \uparrow \quad \text{otherwise}}}$$

$\mu y. f(x,y) = x$ $\quad \in \mathcal{R} \cap Tot$

$\quad \in \mathcal{R}_0$

Exercise :



$K_1, \dots, K_n < K$

→ Does this process terminate ? Why ?