FINANCIAL MATHEMATICS

INTRODUCTION

Financial Mathematics connections within this class:

- **Risks**: Assess financial impacts of potential risks
- **Funding and Investment:** Understanding the financial implications of water-related hazards helps governments and organizations attract funding for mitigation projects. Financial models can demonstrate the economic benefits of investing in resilience strategies.
- **Scenario Analysis:** Financial mathematics allows for scenario analysis, where different future conditions (e.g., climate change impacts) can be modelled to assess their financial implications. This helps in developing strategies to adapt to potential changes in water-related risks.
- **Risk Transfer Mechanisms:** Tools such as catastrophe bonds and derivatives are used to transfer risks associated with water-related hazards. Financial mathematics underpins the valuation and pricing of these instruments, allowing for better risk management.
- **Portfolio Optimization:** In the context of urban planning and resource management, financial mathematics can help optimize resource allocation among various projects aimed at reducing vulnerability to water-related hazards.

INTRODUCTION

Financial Mathematics constitutes an operational tool that allows us **to compare money flows deferred over time.** In fact, it allows you to:

- correctly compare the quality of a capital investment (stock) against the analysis of its revenues (flows);
- compare the benefits and costs linked to the purchase of a good, which occur at different timescales;
- evaluate the profitability of an investment against a series of future revenues generated by the investment itself.

FINANCIAL OPERATIONS

Financial services are all those operations that involve the exchange between services referring to different periods, that is, they are movements of money, between two or more economic entities in a defined time horizon.

Financial Operations

Suppose you have to choose between the following pairs of benefits:

A. 5€ at time 2 or 5€ at time 2

B. 5€ at time 2 or 5€ at time 3

C. 6€ at time 2 or 5€ at time 2

D. 5€ at time 2 or 6€ at time 3

The two performances are indifferent.

The first service (5€ at time 2) is preferable.

The first service (6€ at time 2) is preferable.

It depends.

Interest rate

- **Money** has a **time value**
- Individuals and entrepreneurs borrow money from friends, banks, financial institutions or fellow businessmen
- These borrowings have to be repaid with **interest**
- Just as any resource is available to a person at a price, similarly **interest is the price for acquiring money from other**

Interest rate

The interest rate is the cost of utilizing capital, calculated as a percentage of the borrowed capital.

The unit of measurement for interest is the interest rate (r), The interest rate is defined as the interest accrued on a unit of currency over a specified time period. It can be expressed in percentage terms (e.g., r = 5%) or in decimal form (e.g., r = 0.05).

The interest rate is influenced by:

- **Risk**: There is a direct positive correlation between risk and interest rate (higher risk leads to a higher interest rate).
- **Investment Duration**: A longer investment duration typically results in a higher interest rate.

Interest rate

- An interest rate is the amount of interest due per period, as a proportion of the amount lent, deposited, or borrowed
- It is the reward to the person who is not spending the amount of money for herself/himself
- Such a person has made an investment of money in order to get a reward
- To her/him this **interest** is nothing but the **return on investment**

Interest rate

- The relative return of risk-free securities (bonds or bank account) is called the interest rate
- It is the return that can be achieved in a risk-free investment
- Consider a pure discount bond that sells today at a price **P(0)** and matures with a nominal payment **P(T)**. The bond's ***interest rate*** (in a single-period model) is the value ***r*** that solves:

$$P(0)(1+r)=P(T)$$

$$r = [P(T)-P(0)]/P(0)$$

Interest rate

NOMENCLATURE		
P(0) or C_0	Present Value (Principal or Capital (≅))	The present value is the current value of future cash flows
P(T) or C_n	Future Value or Final Balance (≅)	Amount to which an investment will grow after earning interest
r	Interest rate per compound period	The cost of debt for the borrower and the rate of return for the lender
n	The number of periods compound	
Df	Discount Factor	Present value of a €1 future payment

Interest rate

SINGLE-PERIOD MODEL

$0 < n \leq 1$ year

Simple Interest

Interest earned only on the original investment (interest paid on the interest received)

The interest earned on a capital $\mathbf{C_0}$ lent over a period n at a rate r is

$$nrC_0$$

MULTI-PERIOD MODEL

$n \geq 1$ year

Compound Interest

Interest earned on interest (interest not paid on the interest received)

A capital $\mathbf{C_0}$ lent over a period n at a rate r grows to

$$(1 + r)^n C_0$$

SIMPLE Interest rate

Simple interest

r = annual interest rate, quoted per annum

$I = C_0 r$ 1 year

$I_t = C_0 rt$ t = months/12 months or days/365 or 360 days

$C_t = C_0 + I_t = C_0(1+rt)$

$C_1 = C_0 + I = C_0(1+r)$

$C_{6m} = C_0 + I_{6m} = C_0(1+0.5r)$ t = 6months/12

COMPOUND Interest rate

Compound interest

r = annual interest rate, quoted per annum

Frequency of the interest rate: 1 year

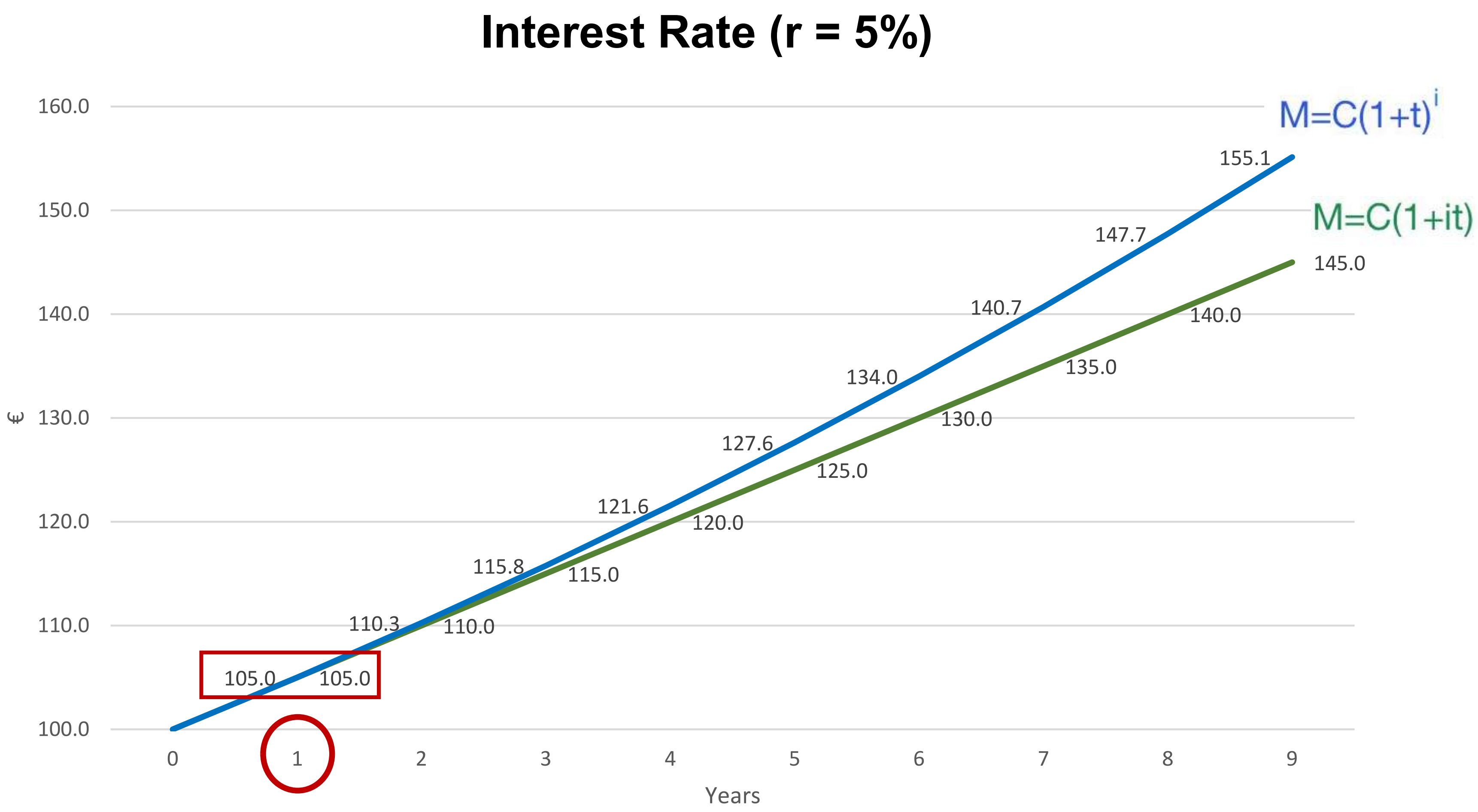
$I = C_0 r$

$C_1 = C_0 + I = C_0(1+r)$

$C_2 = C_1 + I = C_0(1+r)^2$

$C_n = C_{n-1} + I = C_0(1+r)^n$ n = years

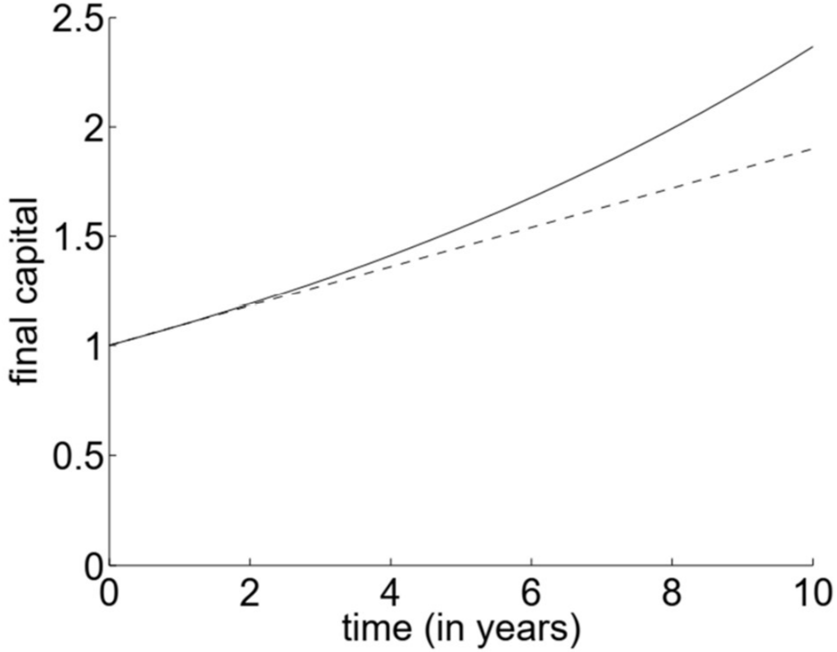
SIMPLE Interest rate vs COMPOUND Interest rate



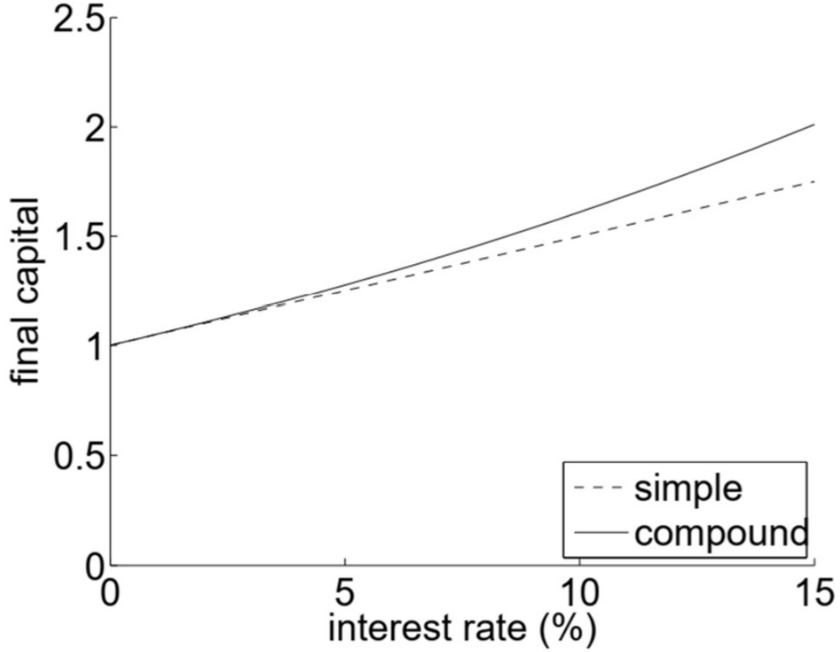
Simple Interest **Compound Interest**

SIMPLE Interest rate vs COMPOUND Interest rate

Comparison of simple interest (dashed line) and compound interest (solid line)



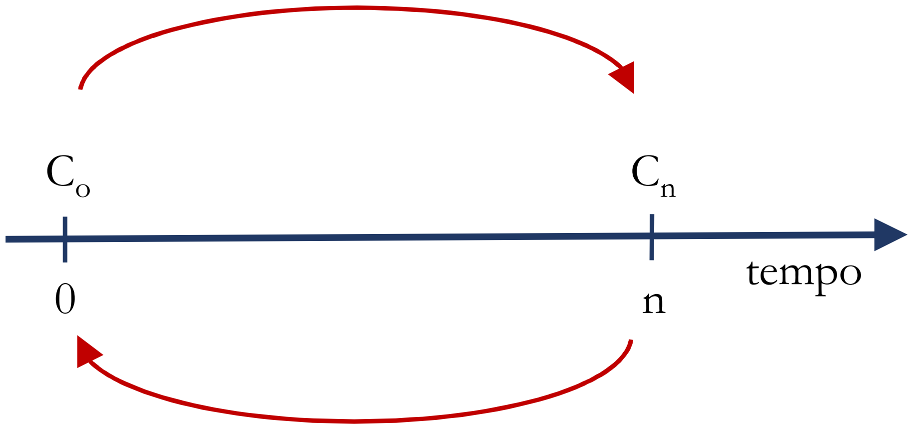
Capital growth in time at a rate of 9%.



Amount of capital after **5 years** for various interest rates

SIMPLE Interest rate: HP: n ≤ 1 year

Interest	$I = C_0 \bullet r \bullet n$
Future Value	$C_n = C_0 + I = C_0 + C_0 \bullet r \bullet n = C_0 \bullet (1 + r \bullet n)$
Present Value	$C_0 = \dfrac{C_n}{(1+r \cdot n)}$



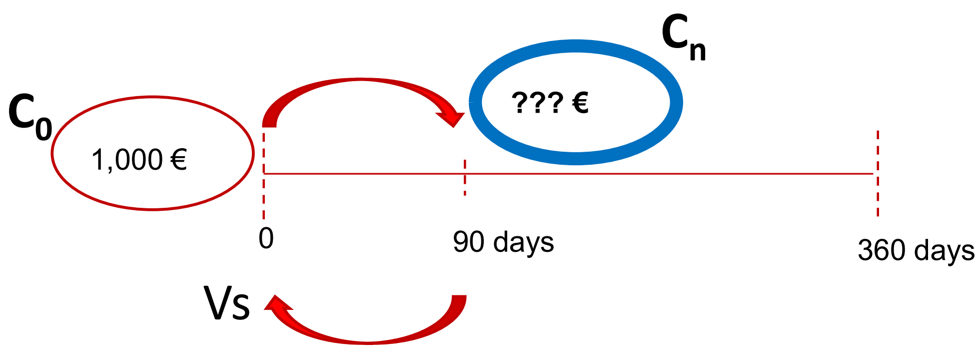
SIMPLE Interest rate: HP: $n \leq 1$ year

Example (Exercise)

The sum of € 1,000 is deposited in the bank at a simple interest of 6%.You want to know the amount, after 90 days of:

a) **interest;**
b) **final value**;
c) **Present value.**

Consider the 360-day business year.



n $n = 90/360 = 0.25$

Interest $I = C_0 \cdot r \cdot n = 1{,}000 \cdot 0.06 \cdot (0.25) = 15$ €

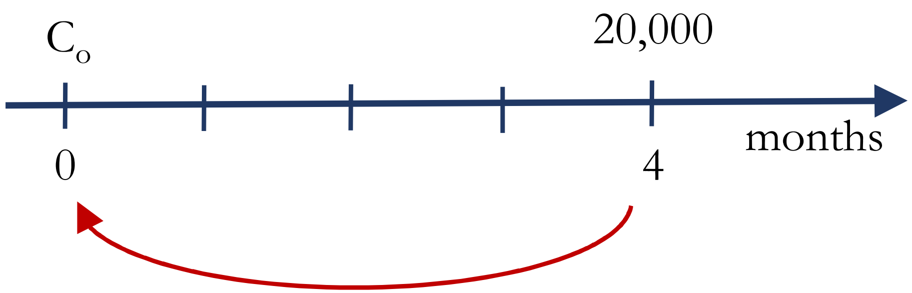
Final value $C_n = C_0 + I = C_0 + C_0 \cdot r \cdot n = C_0 \cdot (1 + r \cdot n) =$

$= 1{,}000 \cdot (1 + 0.06 \cdot 0.25) = 1{,}015$ €

Present Value $C_0 = \frac{C_n}{1+r\cdot n} = \frac{1{,}015}{1+(0.06\cdot 0.25)} = 1{,}000$ €

Example (Exercise)

Adele has to pay off a debt of €20,000 in 4 months. How much should she pay to get out of her debt immediately with a discount rate of 6%?



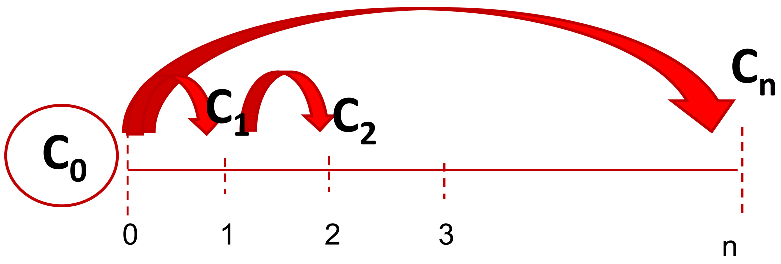
$$C_0 = \frac{C_n}{1 + r \cdot n} = \frac{20{,}000}{1 + 0.06 \cdot \frac{4}{12}} = 19{,}607\ €$$

COMPOUND Interest rate: HP: n > di 1 year

Interest $I = C_n - C_0 = C_0 \cdot q^n - C_0 = C_0 \cdot (q^n - 1)$

Future Value $C_n = C_0 \cdot q^n$

Present Value $C_0 = \frac{C_n}{q^n}$



Where: $q = 1 + r$

COMPOUND Interest rate: HP: n > di 1 year

Exercise 1

How much will the capital of 1,000 € (C_0) invested in securities at a rate of 7% amount to in 10 years (*n*)?

$$C_n = C_0 \times q^n = 1{,}000 \times (1+0.07)^{10} = 1{,}967 \text{ €}$$

If the interest is not compounded, would the amount be higher or lower?

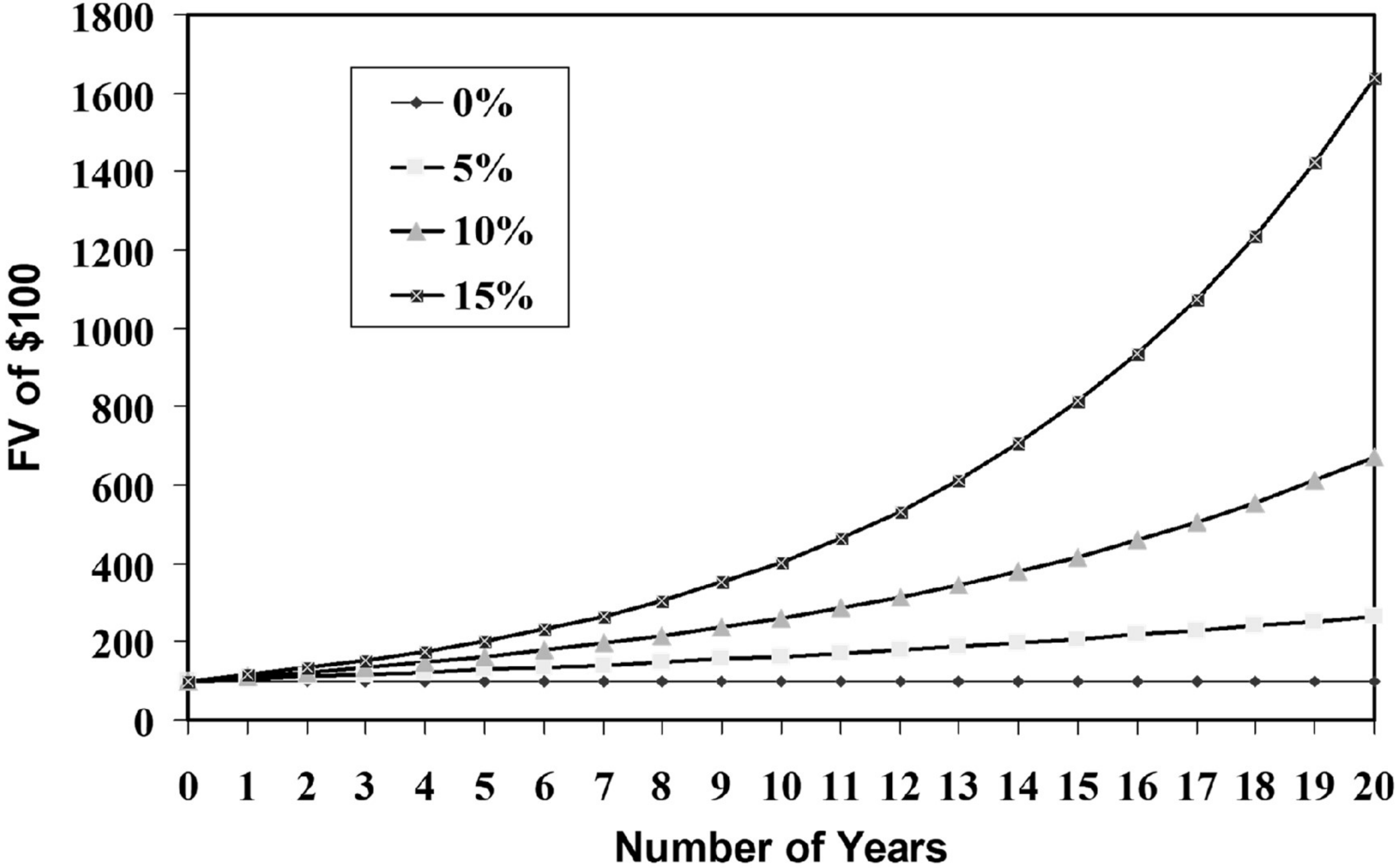
LOWER

Exercise 2

By depositing the sum of €1000 in the bank today at an interest rate of 5%. What will be the interest accrued after eight years?

$$I = C_0 \times (q^n - 1) = 1{,}000 \times ((1+0.05)^8 - 1) = 477 \text{ €}$$

COMPOUND Interest rate: HP: n > di 1 year



Future Value: SIMPLE vs COMPOUND Interest

Example: Interest earned at a rate of 6% for five years on a principal balance of $100

SIMPLE INTEREST

Interest earned at a rate of 6% for five years on a principal balance of $100

Today	Future Years				
	1	2	3	4	5
Interest Earned Per Year	6	6	6	6	6
100	106	112	118	124	**130**

Interest Earned Per Year = $100 x 0.06 =$ 6

Value at the end of Year 5 = **$130**

COMPOUND INTEREST

Interest earned at a rate of 6% for five years on the previous year's balance

Today	Future Years				
	1	2	3	4	5
Interest Earned Per Year	6	6.36	6.74	7.15	7.57
100	106	112.36	119.10	126.25	**133.82**

Value at the end of Year 5 = **$133.82**

= FV = $100 x $(1+0.06)^5$ = $133.82

Δ= 2.94%

COMPOUND Interest

How does a million euros vary with time and interest rates?

Rate/Yrs	1 yr	5 yrs	10 yrs	20 yrs
1%	990,099	951,466	905,287	819,544
2%	980,392	905,731	820,348	672,971
3%	970,874	862,609	744,094	553,676
4%	961,538	821,927	675,564	456,387
5%	952,381	783,526	613,913	376,889
6%	943,396	747,258	558,395	311,805
7%	934,579	712,986	508,349	258,419
8%	925,926	680,583	463,193	214,548
9%	917,431	649,931	422,411	178,431
10%	909,091	620,921	385,543	148,644

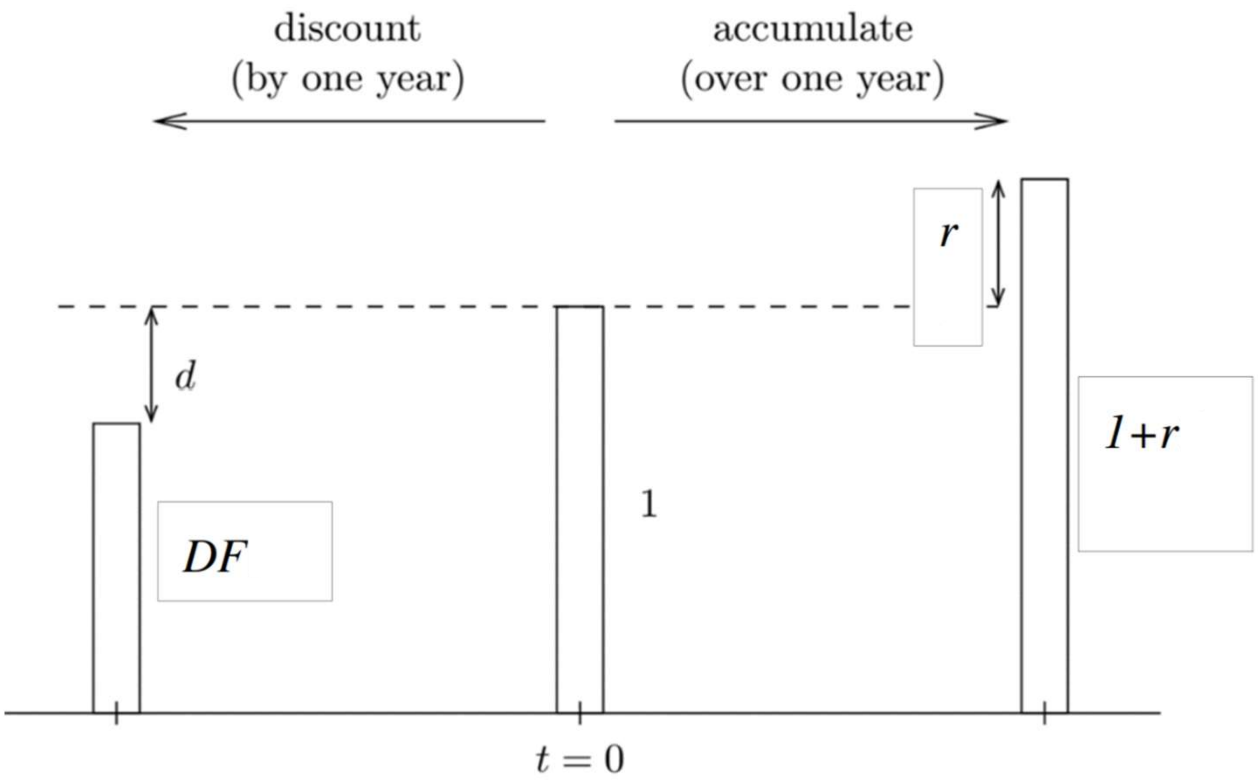
$$C_0 = \frac{C_n}{q^n}$$

As time increases, value decreases

As rate increases, value decreases

DISCOUNTING

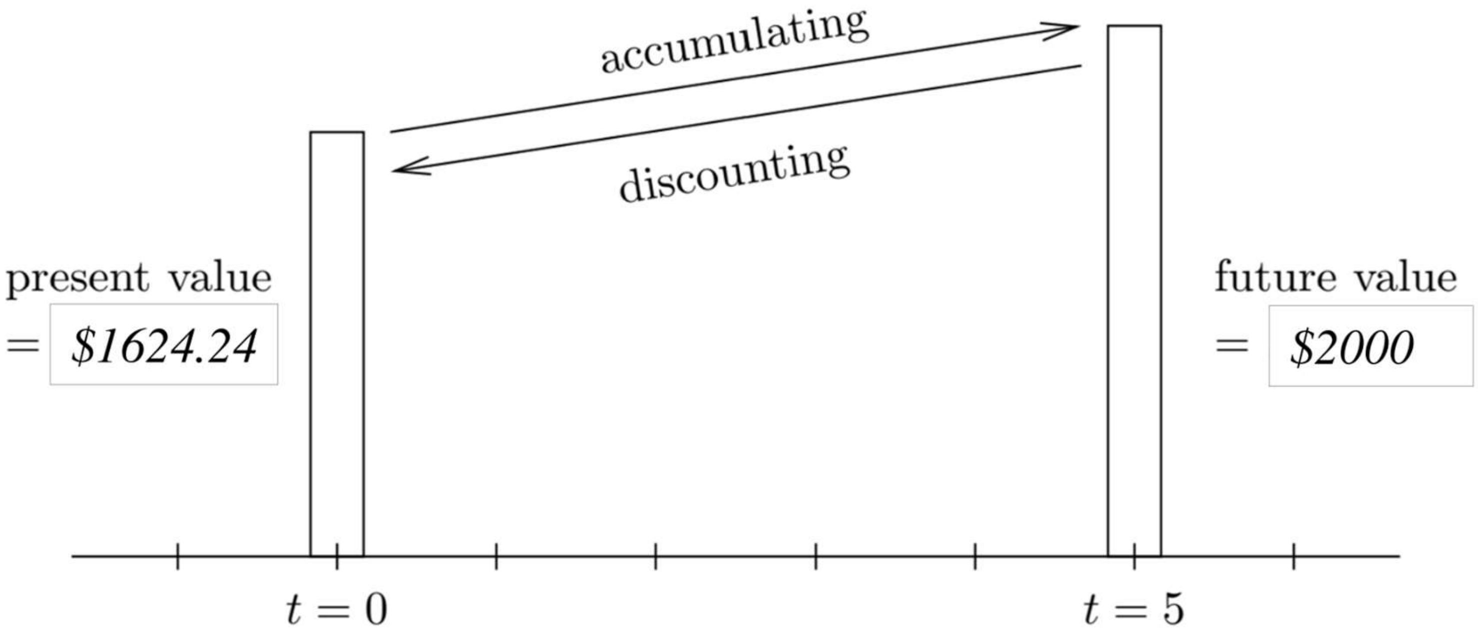
When you move a payment **forward** in time, it **accumulates**
When you move it **backward**, it is **discounted**



DISCOUNTING

The time value of money

\$1624.24 now is equivalent to \$2,000 in five years at a rate of 4.25%



PRESENT VALUES

Present value can be defined as follows:

$$PV = \frac{FV_t}{(1+r)^t}$$

Example
You just bought a new computer for $3,000. The payment terms are 2 years same as cash. If you can earn 8% on your money, how much money should you set aside today in order to make the payment when due in two years?

$$PV = \frac{3{,}000}{(1.08)^2} = \$2{,}572$$

PRESENT VALUES

Discount Factor (DF) = PV of $1

Discount Factors can be used to compute the present value of any cash flow

$$DF = \frac{1}{(1+r)^t}$$

$$PV = FV \frac{1}{(1+r)^t}$$

Given any variables in the equation, you can solve for the remaining variable

ANNUITIES

- An **annuity is a sequence of FIXED PAYMENTS with FIXED FREQUENCY**
- The term “annuity“ originally referred to annual payments (hence the name), but it is now also used for payments with any frequency
- Annuities appear in many situations: for instance, interest payments on an investment can be considered as an annuity
- An important application is the schedule of payments to pay off a loan

ANNUITIES

Annuities can be:

- An annuity with a fixed number of payments is called an **annuity certain (ordinary)**
- An annuity whose number of payments depend on some other event is a **contingent annuity**

- **Annuities in arrear (Postponed) or Annuities Due (in advance or immediate),** based on the expiry of each annuity, respectively if they occur at the end or beginning of the year.

- **Limited or Unlimited**, based on the overall duration of the series of benefits. (It is assumed that it is unlimited when it repeats for more than 80 years).

ANNUITIES

1. Limited postponed

2. Unlimited postponed

3. Limited immediate

4. Unlimited immediate

LIMITED POSTPONED ANNUITIES

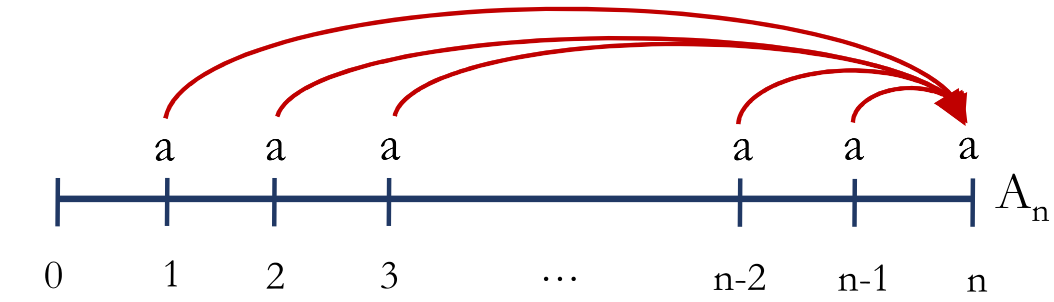
1. Limited postponed

2. Unlimited postponed

3. Limited immediate

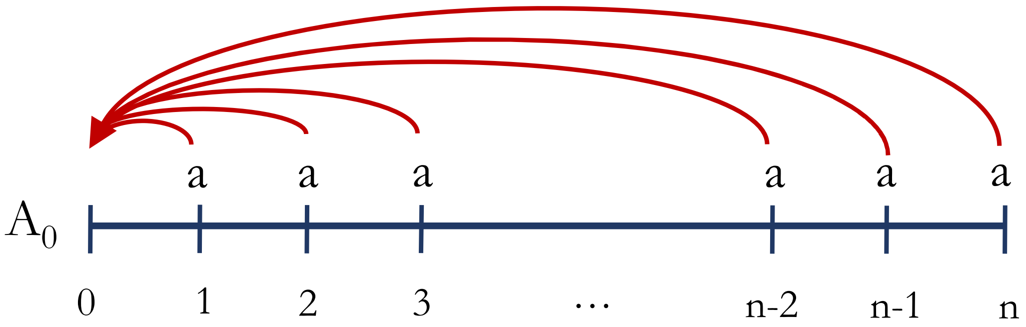
4. Unlimited immediate

LIMITED POSTPONED ANNUITIES



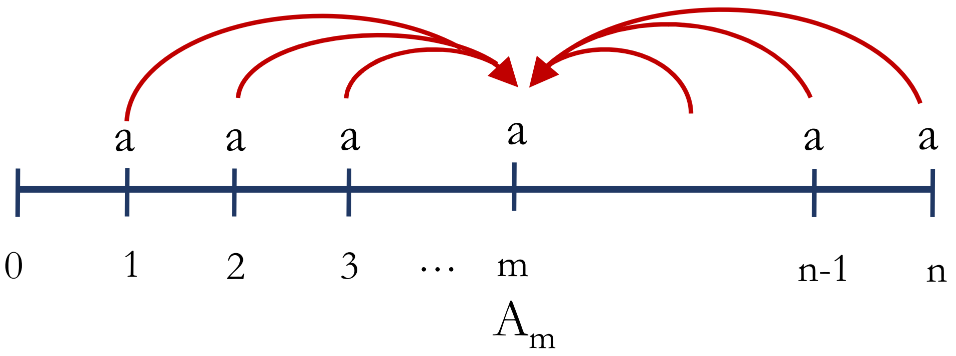
FUTURE VALUE

$$A_n = a\frac{(q^{n-1}\cdot q-1)}{q-1} = a\frac{q^n-1}{r}$$



PRESENT VALUE

$$A_0 = A_n \cdot \frac{1}{q^n} = a\frac{q^n-1}{r\cdot q^n}$$



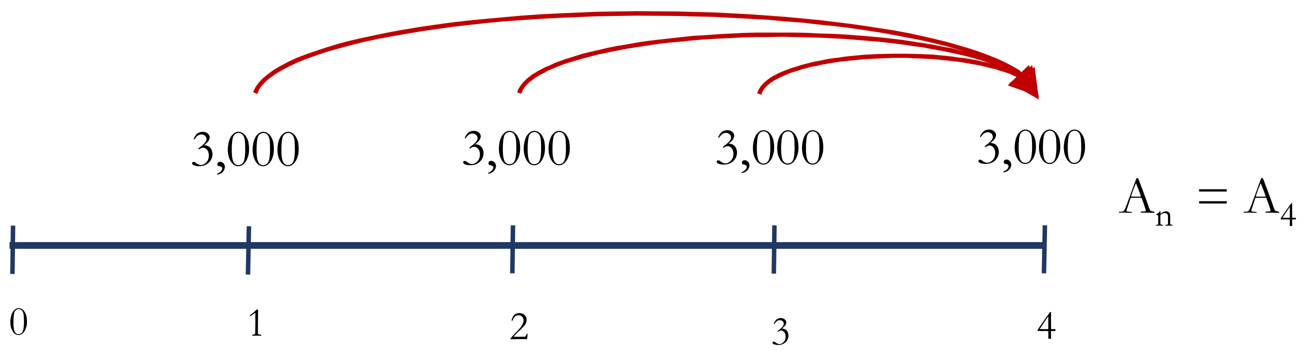
VALUE at m

$$A_m = A_0 \cdot q^m = a\frac{q^n-1}{r\cdot q^n}\, q^m$$

LIMITED POSTPONED ANNUITIES

EXERCISE

From today, by right, at the end of each year and for 4 consecutive years, you have an income of €3,000. Determine the total amount that can be withdrawn at the end of the period with a discount rate = 3%



$$A_4 = 3{,}000\ \frac{(1+0.03)^4 - 1}{0.03} = 12{,}550.88\ €$$

LIMITED IMMEDIATE ANNUITIES

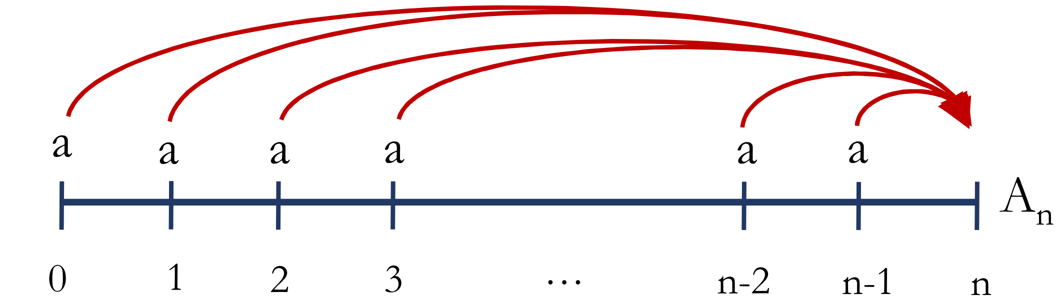
1. Limited postponed

2. Unlimited postponed

3. Limited immediate

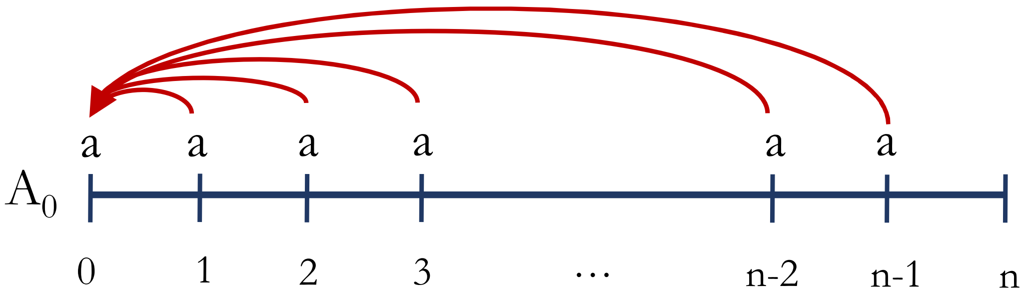
4. Unlimited immediate

LIMITED IMMEDIATE ANNUITIES



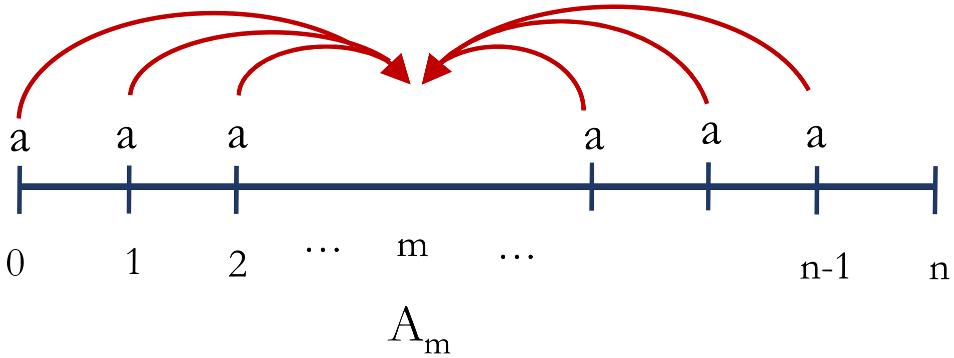
FUTURE VALUE

$$A_n = q\,a\,\frac{q^n - 1}{r}$$



PRESENT VALUE

$$A_0 = q\,a\,\frac{q^n - 1}{r \cdot q^n}$$



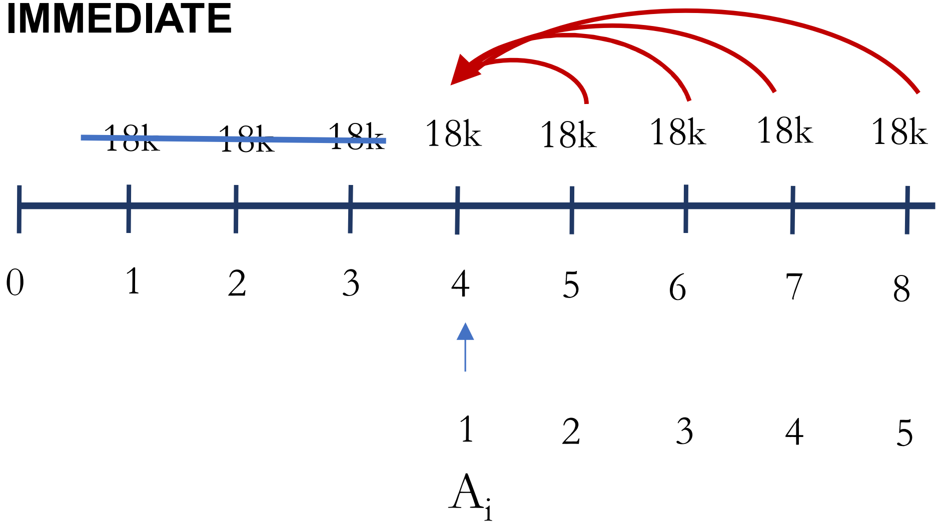
VALUE at m

$$A_m = A_0 \cdot q^m = q\,a\,\frac{q^n - 1}{r \cdot q^n}\,q^m$$

ANNUITIES

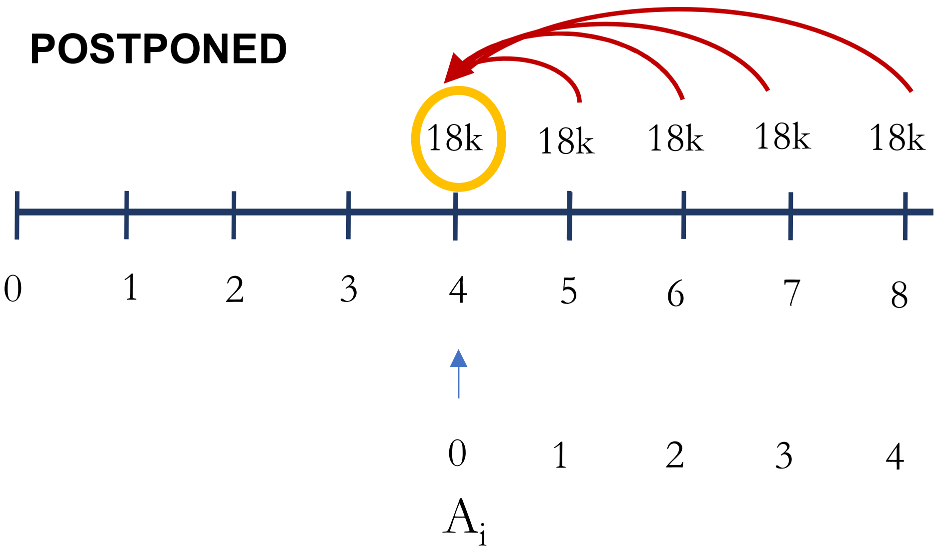
EXERCISE: The supply of a system is paid in eight fixed annual installments in arrears of €18,000 each. After paying the third installment, on the occasion of the expiry of the fourth installment you want to set aside the entire capital to pay off the debt in the remaining years. How much should I set aside if the interest rate on my capital is 6%?

IMMEDIATE



$$A_i = 1.06 \cdot 18{,}000 \cdot \frac{1.06^5 - 1}{0.06 \cdot 1.06^5} = 80{,}372 \text{ €}$$

POSTPONED



$$A_{i-1} = 18{,}000 \cdot \frac{1.06^4 - 1}{0.06 \cdot 1.06^4} = 62{,}372 \text{ €}$$

$$A_i = 18{,}000 + 62{,}372 = 80{,}372 \text{ €}$$

ANNUITIES

Equity Capital	-	Debt Capital
ECONOMIC	-	FINANCIAL

UNLIMITED POSTPONED ANNUITIES

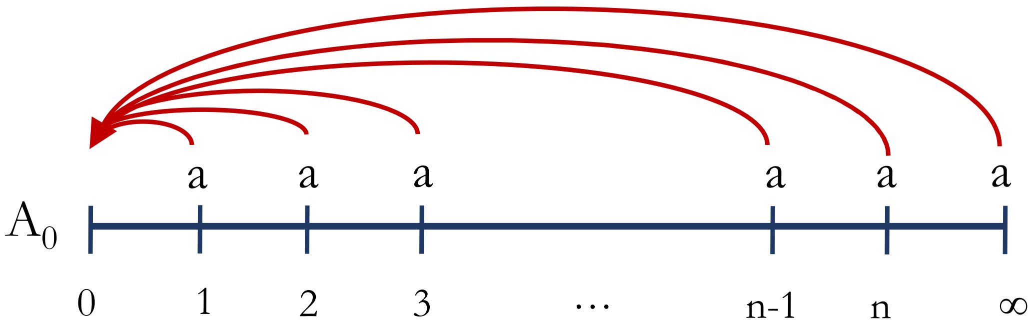
1. Limited postponed

2. Unlimited postponed

3. Limited immediate

4. Unlimited immediate

UNLIMITED POSTPONED ANNUITIES



PRESENT VALUE

$$A_0 = a \frac{q^n - 1}{r\, q^n} \longrightarrow A_0 = a \frac{q^\infty - 1}{r\, q^\infty} \cong \frac{a\, q^\infty}{r\, q^\infty} \longrightarrow A_0 = \frac{a}{r}$$

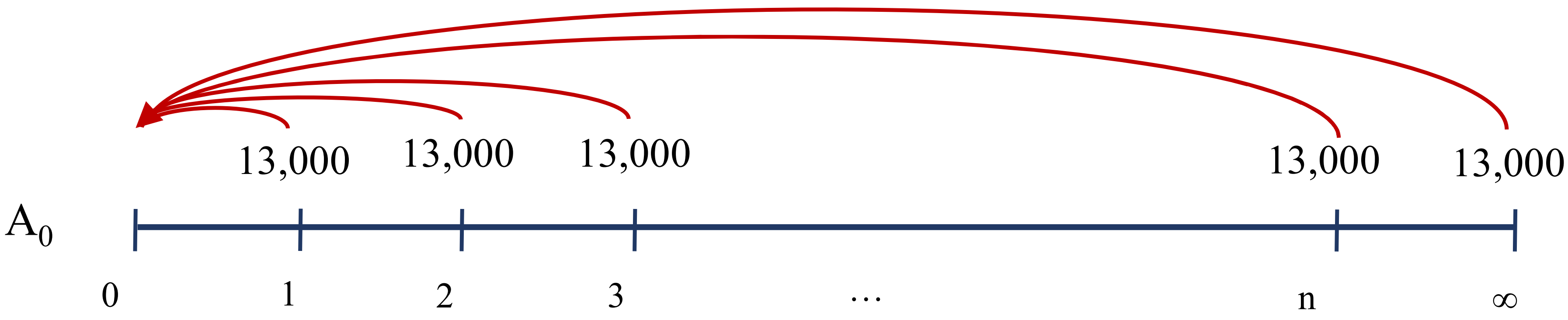
LIMITED POSTPONED ANNUITIES

UNLIMITED POSTPONED ANNUITIES

EXERCISE

A sewage treatment plant estimates that it annually accumulates damages from interruption of business due to malfunctioning of machinery and subsequent slowdown of production equal to €10,000. The average total annual deferred expenses incurred by the plant to repair the machinery amount to €3,000. What will be the value of the development of a technology that allows to avoid the blockage of the machinery (value of the avoided damage) assuming a continuous duration of the production cycle and a capitalization rate of 4%?

$$Rn = 10{,}000 + 3{,}000 = 13{,}000$$



$$A_0 = \frac{13{,}00}{0.04} = 325{,}000\ €$$

UNLIMITED IMMEDIATE ANNUITIES

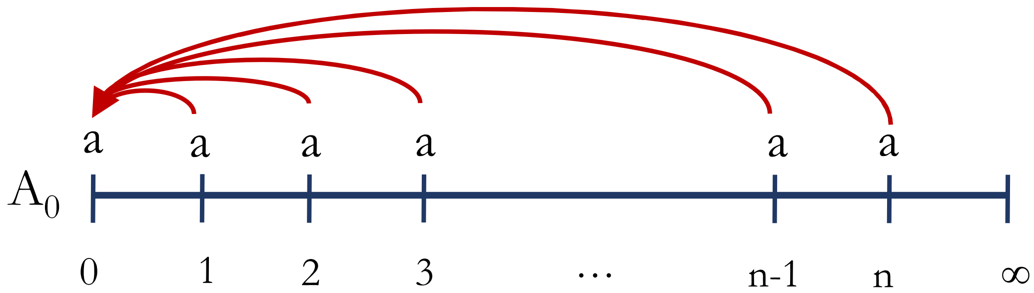
1. Limited postponed

2. Unlimited postponed

3. Limited immediate

4. Unlimited immediate

UNLIMITED IMMEDIATE ANNUITIES



PRESENT VALUE

$$A_0 = q\frac{a}{r}$$

SINKING FUND PAYMENT

"SINKING FUND PAYMENT" OR "ANNUITY PAYMENT"

Sum of money that is accumulated for a certain number of years for the purpose of building/renewing capital for a specific purpose.

$$a = A_n \frac{r}{q^n - 1}$$

EXERCISE:

A company purchases a machine for the price of €30,000. Assuming an economic duration of 12 years and a residual value at the 12th year equal to €5,000. How much will the company have to set aside annually to have the sum needed to purchase the same machinery at the same price after 12 years at a rate = 5%?

$$a = 25{,}000 \frac{0.05}{(1+0.05)^{12} - 1} = 1{,}570 \text{ €}$$

INSTALLMENT RATE

The Installment Amount (I_A) is that sum of money that must be paid to extinguish, in a certain number of years (n), a debt initially contracted,

$$a = A_0 \frac{r\, q^n}{q^n - 1} = I_A$$

$$I_A = P + Int$$

where:

- P = Principal Amount/Portion: repayment of part of the capital borrowed;
- Int = Interest Amount/Portion: payment of interest, calculated progressively on the residual capital to be paid.

Amortization Plan" or "Repayment Schedule

AMORTIZATION PLAN

It is a table that shows the progression of the amortization share, meaning the principal portion and the interest portion. The amortization plan consists of several columns arranged from left to right in the following order:

1. Installment/Year Number
2. Constant Annual Installment Amount
3. Interest Portion
4. Principal Portion
5. Paid-Off Debt
6. Remaining Debt

In the case of deferred constant annual installments, the amortization plan will be:

1	2 (Fixed)	4	3	5	6
Year	**Constant Annual Installment Amount (I_A)**	**Principal Portion (P_p)**	**Interest Portion (I_P)**	**Paid-Off Debt (PoD)**	**Remaining Debt (RD)**
n	$I_A = A_0 \times \frac{r\, q^n}{q^n - 1}$	$P_p = I_{A(n)} - I_{P(n)}$	$I_P = RD_{(n-1)} \times r$	$PoD = PoD_{(n-1)} + P_{P(n)}$	$RD = RD_{(n-1)} - P_{P(n)}$

where *n* is the current year, *A0* is the principal to be repaid.

Amortization Plan" or "Repayment Schedule

EXERCISE: I approach a credit institution to obtain a 10-year mortgage loan for a first home in the amount of 100,000 Euros at an interest rate of 2.0%. **Design the amortization plan**.

LOAN	
Loan (A0) - Initial Principal	$ 100,000
Repayment Period (n) - Years	10
Interest Rate (r)	2.00%
Constant Annual Installment Amount (IA)	**?**
q^n	?

1	2	4	3	5	6
n	$I_A = A_0 \times \frac{r\,q^n}{q^n - 1}$	$P_p = I_{A(n)} - I_{P(n)}$	$I_P = RD_{(n-1)} \times r$	$PoD = PoD_{(n-1)} + P_{P(n)}$	$RD = RD_{(n-1)} - P_{P(n)}$

Year	Constant Annual Installment Amount (I_A)	Principal Portion (P_p)	Interest Portion (I_P)	Paid-Off Debt (PoD)	Remaining Debt (RD)
0	$ -	$ -	$ -	$ -	$ 100,000
1	?	?	?	?	?
2	?	?	?	?	?
3	?	?	?	?	?
4	?	?	?	?	?
5	?	?	?	?	?
6	?	?	?	?	?
7	?	?	?	?	?
8	?	?	?	?	?
9	?	?	?	?	?
10	?	?	?	?	?

REFERENCES

- Cvitanić J., Zapatero F. (2004): “Economics and Mathematics of Financial Markets”. The MIT Press
- Niesen J. (2012): “Financial Mathematics I”. University of Leeds