$T_n, T \in \mathcal{D}'(U)$ $\quad T_n \to T$ (as distributions)

if $\forall \phi \in C_c^\infty(U)$ $\quad T_n(\phi) \to T(\phi)$

If $T_n \to T \Rightarrow \forall \alpha \quad D^\alpha T_n \to D^\alpha T$

$$D^\alpha T_n(\phi) = (-1)^{|\alpha|} T_n(D^\alpha \phi) \to (-1)^{|\alpha|} T(D^\alpha \phi)$$
$$= D^\alpha T.$$

Let $f_n, f \in L^p(U)$ $\quad p \in [1, +\infty]$

$f_n \rightharpoonup f$ in L^p $\quad \left(\int_U f_n g \, dx \to \int_U f \cdot g \, dx \quad \forall g \in L^{p'}(U) \right.$

for $p = +\infty$ $\quad f_n \overset{*}{\rightharpoonup} f$ $\qquad\qquad p' = \frac{p}{p-1}$

$\Rightarrow \quad T_{f_n} \to T_f$ (in the sense of distr.)

since $\mathcal{C}_c^\infty(U) \subseteq L^{p'}(U) \quad \forall p' \in [1, +\infty]$

μ_n, μ Radon measure

$$\mu_n \longrightarrow \mu \qquad \int_U g \, d\mu_n \to \int_U g \, d\mu \quad \forall g \in \mathcal{C}_c(U)$$

$$\Downarrow$$

$$T_{\mu_n} \to T_\mu$$

Obs $\quad f_n \longrightarrow f$ a.e. in $U \qquad f_n, f \in L^1_{loc}(U)$

$$f_n = \begin{cases} n\, e^{\frac{1}{|x|^2 - 1}} & |x| < \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases}$$

$$T_{f_n} \longrightarrow T_f$$

$f_n \to 0$ a.e. (in $\mathbb{R}^N \setminus \{0\}$)

$$T_{f_n} \longrightarrow \delta_0$$

Obs $f_n, f \in L^p(U)$ $\quad T_{f_n} \to T_f$ as distrib.

$\|f_n\|_{L^p} \leq C \quad \forall n$

$\Longrightarrow$ for $p \in (1, +\infty]$ $\quad f_n \rightharpoonup f$ in $L^p(U)$

for $p = 1$ NOT TRUE.

$f_n(x) = \chi_{(n, n+1)}(x)$
in $\mathbb{R}$

$\boxed{\text{Product of convolution}}$

$U \subseteq \mathbb{R}^n$

$T \in \mathcal{D}'(U)$ $\quad \underbrace{\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)}$

$\forall x \in \mathbb{R}^n$ $\quad \phi^x(y) := \phi(x-y)$

$V =$ open set in $\mathbb{R}^n$ $= \{ x \in \mathbb{R}^n : x - y \subseteq U \ \ \forall y \in \operatorname{supp}\phi \}$

V depends on ϕ $\quad x \in V \iff x - \operatorname{supp}\phi \subseteq U$

(V can also be empty)

$V \neq \emptyset$ $\quad$ if $x \in V \implies \operatorname{supp}(\phi^x) \subseteq U$

so $\phi^x \in \mathcal{C}_c^\infty(U)$ $\forall x \in U$

We define $\forall x \in U$

$$\underbrace{T * \phi}_{\text{it is a function}} (x) := T(\phi^x)$$

if $T = T_f$ $\quad f \in L^1_{loc}$ $\quad$ $\underset{x \in U}{f * \phi(x)} = \int_U f(y)\, \phi(x-y)\, dy$

$$= \int_U f(y)\, \phi^x(y)\, dy$$

$$= T_f(\phi^x)$$

$$\boxed{\text{Obs 1} \quad T * \phi \in C(U)}$$

$x_n \to x$ in U $\quad \phi^{x_n} \to \phi^x$ in $C_c^\infty(U)$

$$\Rightarrow T(\phi^{x_n}) \to T(\phi^x)$$

2) δ_0 is the unit $\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\delta_0 * \phi(x) = \delta_0(\phi^x) = \phi^x(0) = \phi(x)$$

$$\phi^x(y) = \phi(x-y)$$

3) $\forall \alpha \in \mathbb{N}^n$

$$D^\alpha[(T*\phi)](x) = [(D^\alpha T) * \phi](x) =$$
$$= [T * (D^\alpha \phi)](x)$$

one deduces that $\boxed{T*\phi \in \mathcal{C}^\infty(V)}$

proof $x \in V$ e_i $i = 1 \dots n$

$x + te_i \in V$ for $|t| \leq t_0$

$t \in \mathbb{R}$

$T(\phi^x) = T * \phi(x)$

in $\mathcal{E}^\partial_c(U)$

$$\underbrace{\frac{T(\phi^{x+te_i}(\cdot)) - T(\phi^x(\cdot))}{t}} \xrightarrow[t \to 0]{} \underbrace{T\left(\frac{\partial}{\partial x_i} \phi^x(\cdot)\right)}$$

$$\phi^{x+te_i}(y) = \phi(x+te_i - y) = \phi^x(y - te_i)$$

$$\phi(x - (y - te_i))$$

$$\frac{T * \phi(x+te_i) - T * \phi(x)}{t} \longrightarrow \left(T * \frac{\partial}{\partial x_i}\phi\right)(x)$$

$$\frac{\partial}{\partial x_i}(T * \phi)(x) = \left(T * \frac{\partial}{\partial x_i}\phi\right)(x)$$

$$D^\alpha(T*\phi)(x) = T*(D^\alpha\phi)(x) \qquad \forall \alpha$$

$$(D^\alpha T)*\phi(x) = D^\alpha T(\phi^x) = (-1)^{|\alpha|}\, T\left(D^\alpha_y \phi^x(\cdot)\right) =$$

$$= \cancel{(-1)^{|\alpha|}}\, T\left(\cancel{(-1)^{|\alpha|}}\, D^\alpha_x \phi^x(\cdot)\right)$$

$$\phi^x(y) = \phi(x-y)$$

$$D^\alpha_y \phi^x(y) = (-1)^{|\alpha|} D^\alpha_x \phi(x-y)$$

Appl. If $\mathcal{L}$ is a linear differential operator with constant coefficient and $T \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution for $\mathcal{L}$ $\left(\mathcal{L}(T) = \delta_0\right)$

then $\forall\, \phi \in C_c^\infty(\mathbb{R}^n)$

$u = T * \phi \in C^\infty(\mathbb{R}^n)$ is a classical solution of $\mathcal{L}(u) = \phi$

$\phi \in C_c^\infty(\mathbb{R}^n)$ GIVEN

$$-\Delta u = \phi \text{ in } \mathbb{R}^n \;\xrightarrow[n \geq 3]{}\; u(x) = \int_{\mathbb{R}^n} \frac{\phi(x-y)}{|y|^{n-2}}\, c_n \, dy$$

proof $\phi \in C_c^\infty(U)$ $\quad \underline{\mathcal{J}(T) = \delta_0}$ $\quad$ T fund sol. $\quad$ δ_0 unit.

$$\underbrace{\mathcal{J}(T*\phi)}(x) = \left[\left[\mathcal{J}(T)\right] * \phi\right](x) \overset{T \text{ fund sol.}}{=} \delta_0 * \phi(x) \overset{\delta_0 \text{ unit}}{=} \phi(x)$$

since $\mathcal{J}$ is the linear combination (with constant coefficients) of derivatives D^α... $\square$.

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host property: $T \in \mathcal{D}'(U)$ $\quad \phi \in C_c^\infty(\mathbb{R}^n)$

$\psi \in C_c^\infty(V)$ $\qquad x \in V \Leftrightarrow x - \operatorname{supp}\phi \subseteq U$

$\tilde\phi(x) = \phi(-x)$ $\qquad \tilde\phi * \psi(x) = \int_V \phi(y-x)\psi(y)\,dy \Rightarrow \operatorname{supp}(\tilde\phi * \psi) \subseteq U$

$$\int_V \boxed{(T*\phi)(x)} \cdot \psi(x)\,dx = T(\tilde\phi * \psi)$$

$y \in V \Leftarrow y - \operatorname{supp}\phi \subseteq U$

$x \in U$

$y - x \subseteq \operatorname{supp}\phi$

Coherent

$T = T_f \qquad f \in L^1_{loc}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \underbrace{f * \phi(x)}_{T_f * \phi(x)} \psi(x)\, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)\, \phi(\underbrace{x-y}) \psi(x)\, dy\, dx$$

$$(x,y) \longmapsto (z,y) \qquad z = -(x-y) \qquad x = y - z$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y) \underbrace{\phi(-z)\, \psi(y-z)\, dz}_{= \tilde{\phi}(z)}\, dy \;=\; \text{Fubini Tonelli}$$

$$= \int_{\mathbb{R}^n} f(y) \cdot [\tilde{\phi} * \psi](y)\, dy = T_f(\tilde{\phi} * \psi)$$

Theorem $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$

$\left(\forall T \in \mathcal{D}'(U) \quad \exists \phi_n \in C_c^\infty(U)\right.$ such that

$T_{\phi_n} \longrightarrow T$ in the sense of distributions$\left.\right)$

$\forall \psi \in C_c^\infty(U) \qquad \int_U \phi_n(x)\psi(x)\,dx \longrightarrow T(\psi)$

Remember

$$\eta(x) = \begin{cases} \underline{\underline{C}} \cdot e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$$

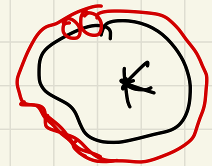
$$\int_{\mathbb{R}^n} \eta(x)\,dx = 1 \qquad \operatorname{supp} \eta = B(0,1) \qquad \eta(x) = \bar{\eta}(|x|)$$

$\forall\, t>0$

$$\eta_t(x) = \frac{1}{t^n}\,\eta\left(\frac{x}{t}\right) \qquad \operatorname{supp} \eta_t \subseteq B(0,t)$$

Let $\phi \in C_c^\infty(U)$

For $t \to 0$ $\quad t>0$



$K = \text{supp}\,\phi$

$$\phi * \eta_t \longrightarrow \phi \quad \text{in } C_c^\infty(U)$$

$\forall\, U'$ open neighborhood of the support of ϕ

$\exists\, t_0 > 0$ such that for $0 < t < t_0$, $\text{supp}(\phi * \eta_t) \subseteq U'$

$$\left(\text{supp}(\phi * \eta_t) = \text{supp}\,\phi + B(0,t) \right)$$

proof Fix $T \in \mathcal{D}'(U)$

I want to reduce to the case T with compact support

$$\underbrace{U = \bigcup_{i=1}^{\infty} V_i}$$

$V_i \subseteq U$

V_i open, bdd, $\overline{V_i}$ compact

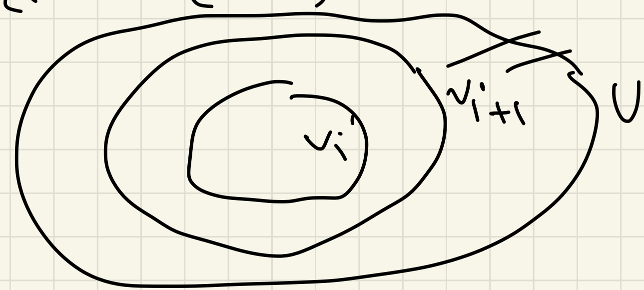
$\overline{V_i} \subseteq V_{i+1}$

$\forall i$

$\xi_i \in C_c^\infty(U)$ such that $\xi_i \equiv 1$ on $\overline{V_i}$

$\xi_i \equiv 0$ $U \setminus V_{i+1}$

$0 \le \xi_i \le 1$ everywhere



$T_i = \xi_i T \in \mathcal{D}'(U)$

$\forall \phi \in C_c^\infty(U)$

$T_i(\phi) := T(\xi_i \phi)$

supp $T_i \subseteq V_{i+1}$

$T_i \to T$ in the sense of distr.

$$T_i(\phi) = T(\underline{\underline{\xi_i \phi}}) \longrightarrow T(\underline{\underline{\phi}}) \qquad \forall \phi \in C_c^\infty(U)$$

$\forall \phi \in C_c^\infty(U)$ $\qquad U = \bigcup_i V_i$

$\exists K$ such that $\quad \operatorname{supp} \phi \subseteq V_{K+1}$

$$\xi_i \phi = \phi \quad \forall i \geq K+1$$

so we are reduced to consider T_i

T_i is a distribution with compact support in U

$(T_i * \eta_t) \longrightarrow T_i$ in the sense of dist.

$T_{T_i * \eta_t} \longrightarrow T_i$

$\boxed{T_i * \eta_t \in C_c^\infty(U) \quad \text{for } t \text{ suff. small}}$