Functions Theory. List of possible questions for the exams

In the following "motivate your answer" requires some explanation (not necessarily the rigorous proof with all the details).

(1) State the definition of Hausdorff measure. Discuss the main properties of Hausdorff measures, in particular the definition of Hausdorff dimension.
Motivate (at least some of) your answers.
Give the definition of k-countable rectifiable and k-rectifiable set.

(2) Recall the definition of distribution. Define the order of a distribution. Give some examples of distributions of order 0 and positive order.
Let $U \subseteq \mathbb{R}^n$ be an open set and $T \in \mathcal{D}'(U)$, be a positive distribution: that is $T(f) \geq 0$ if $f \geq 0$. Which is the order of T? Motivate your answer.

(3) Give the definition of support of a distribution.
Let $T \in \mathcal{D}'(U)$ be a distribution of compact support and $\phi \in C^\infty_c(U)$ such that $\phi \equiv 0$ on the support of T. Is it true that $T(\phi) = 0$? Motivate your answer.
Is it true that a distribution T with compact support has finite order? Motivate your answer.

(4) Recall the notion of convergence in the space of distributions.
Let $f_n \in L^1_{loc}(U)$ such that $f_n \to f$ almost everywhere in U, for some $f \in L^1_{loc}$. Is it true that $T_{f_n} \to T_f$? (hint recall the classical approximation of the distribution δ_0).
What we may deduce about the convergence of T_{f_n} to T_f if $f_n \to f$ locally in L^1 (so in $L^1(K)$ for every compact set inside U)?

(5) Let I be an open bounded interval in $\mathbb{R}$. Define the Sobolev space $W^{1,p}(I)$ for $p \in [1, +\infty]$ and characterize its elements.
For which $p \in [1, +\infty]$ the space $W^{1,p}(I)$ is compactly embedded in $C^{0,1/2}(I)$ (that is the space of 1/2 Holder continuous functions)? Motivate your answer.

(6) Show that monotone functions $f : \mathbb{R} \to \mathbb{R}$ are locally BV in $\mathbb{R}$.
Which is the characterization of $BV(I)$ for I bounded open interval?

(7) Let $U \subseteq \mathbb{R}^n$ a bounded open set of class C^1. Show that every bounded sequence u_n in $W^{1,p}(U)$ admits a subsequence which converges strongly in $L^p(U)$ to some limit u.
What we may say about the convergence of the gradients ∇u_n? To which spaces does the limit u belong?
What can be said about bounded sequences in $BV(U)$?
Prove that the embedding $W^{1,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is never compact, for any $p \in [1, +\infty]$.

(8) Let $U \subseteq \mathbb{R}^n$ be a bounded open set of class C^1. Consider the closed ball

$$B = \{u \in W^{1,p}(U), \|u\|_{W^{1,p}} \leq 1\}.$$

In which functional spaces is this set compact? Motivate your answer.
Let $p = 1$. Is B closed in L^1?

(9) Consider the closed ball

$$B = \{u \in W^{1,p}(\mathbb{R}^n), \|u\|_{W^{1,p}} \leq 1\}.$$

In which functional spaces $L^q(\mathbb{R}^n)$ this set is closed? Motivate your answer.

(10) Let $p > n$: is it true that if $u \in W^{1,p}(\mathbb{R}^n)$ then $u(x) \to 0$ as $|x| \to +\infty$? Motivate your answer.

(11) In which spaces is $W^{1,n}(\mathbb{R}^n)$ continuously embedded? Let $p > n$, in which spaces is $W^{1,p}(\mathbb{R}^n)$ continuously embedded?
Show that $W^{2,n}(\mathbb{R}^n)$ is continuously embedded in L^∞ and the functions in $W^{2,n}(\mathbb{R}^n)$ admit a continuous representative in $C^{0,\alpha}(\mathbb{R}^n)$ for every $\alpha \in (0, 1)$.

(12) Let $U \subseteq \mathbb{R}^n$ be an open bounded set and let $p > 1$. Show that the energy

$$\int_U |Du|^p dx$$

admits a minimizer in the set $K = \{u \in W_0^{1,p}(U), \|u\|_{L^p(U)} = 1\}$. Is the minimizer unique? What can be said in the case $p = 1$?

(13) Let $u \in W^{1,\infty}(\mathbb{R}^n)$ with compact support. Is it true that u is Lipschitz continuous? Motivate your answer.
Is it true that every Lipschitz continuous function in $\mathbb{R}^n$ belongs to $W^{1,\infty}(\mathbb{R}^n)$? Motivate your answer.

(14) Let U be a bounded open set of class C^1 and let $g \in L^p(U)$. Consider the energy

$$E(u) = \int_U |Du|^p + |g - u|^p dx.$$

Show that this energy admits a minimizer among $u \in W^{1,p}(U)$. Motivate your answer.
Deduce the Euler Lagrange equation satisfied by minimizers for $p = 2$.

(15) State the Poincaré inequality for $u \in W^{1,p}(U)$, with $U \subseteq \mathbb{R}^n$. Which are the conditions to be imposed on U? Give a proof of the inequality.
If $u \in W_0^{1,p}(U)$ is a similar inequality valid? Under which conditions on U? Motivate your answer.
What can be said in the case of dimension 1?

(16) Let U be a bounded connected open set of class C^1, and $g \in L^2(U)$, with $\int_U g(x)dx = 0$. Consider the energy

$$E(u) = \int_U \frac{|Du|^2}{2} + g(x)\ u\ dx.$$

Show that E admits a minimum in the closed set $\{u \in W^{1,2}(U), \int_U u(x) = 0\}$. Show that the minimizer is unique.

(17) State the Gagliardo Nirenberg Sobolev inequality in $\mathbb{R}^n$, and some of its corollaries.
Show the relation between the GNS inequality and the isoperimetric inequality for sets of finite perimeter.

(18) Let U be a bounded open set of class C^1 and let $g \in L^1(U)$. Consider the energy

$$E(u) = |Du|(U) + \int_U \sqrt{1 + (g - u)^2} dx.$$

Show that this energy admits a minimizer among $u \in BV(U)$. Motivate your answer.

(19) Let U be a bounded open set of class C^1 and let $E \subseteq \mathbb{R}^n$ a set of finite perimeter. Show that there exists at least one function with minimal total variation in the set $\{u \in BV(\mathbb{R}^n)$, with $0 \le u \le 1$ and such that $u = \chi_E$ in $\mathbb{R}^n \setminus U\}$. Motivate your answer.