$\mathcal{J}$ linear differential operator of order k $\quad \mathcal{J}(u) = \sum_{|\alpha| \leq k} c_\alpha D^\alpha u$

$T \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution of $\mathcal{J}$ if

$\mathcal{J}(T) = \delta_0$

$\mathcal{J}(T) = \sum_{|\alpha| \leq k} c_\alpha D^\alpha T$

$n \geq 3$

$C_n = -\dfrac{1}{n(n-2)\,\omega_n}$ $\qquad C_2 = -\dfrac{1}{2\pi}$

Ex

$\dfrac{C_n}{|x|^{n-2}}$ is a fundamental solution of $\mathcal{J}(u) = -\Delta u$ in $\mathbb{R}^n$ $\forall n \geq 3$

$C_2 \lg|x|$ is a fundamental sol. of $\mathcal{J}(u) = -\Delta u$ in $\mathbb{R}^2$.

$f = \dfrac{C_n}{|x|^{n-2}} \in L^1_{loc}(\mathbb{R}^n)$ since $\dfrac{1}{|x|^\alpha} \in L^1(B(0,R))$ $\iff \alpha < n$

$f = c_2 \lg|x| \in L^1_{loc}(\mathbb{R}^2)$

We have to show that $-\Delta(T_f) = \delta_0 \implies \boxed{\forall \phi \in C^\infty_c(\mathbb{R}^n) \quad T_f(\Delta\phi) = -\phi(0)}$

(*) For $x \neq 0$ $\frac{1}{|x|^{n-2}}$, $\lg|x|$ are smooth functions.

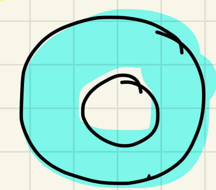
$$D\left(\frac{1}{|x|^{n-2}}\right) = (2-n)\frac{1}{|x|^{n-1}}\frac{x}{|x|} = (2-n)\frac{x}{|x|^n} \qquad D\lg|x| = \frac{1}{|x|}\frac{x}{|x|} = \frac{x}{|x|^2}$$

$$\Delta\left(\frac{1}{|x|^{n-2}}\right) = \operatorname{div}\left(D\left(\frac{1}{|x|^{n-2}}\right)\right) = (2-n)\operatorname{div}\left(\frac{x}{|x|^n}\right) = (2-n)\left[\frac{n}{|x|^n} - \frac{n}{|x|^{n+1}}\sum_i \frac{x_i \cdot x_i}{|x|}\right] =$$

$$= (2-n)\left[\frac{n}{|x|^n} - \frac{n\,|x|^2}{|x|^{n+1}\,|x|}\right] = 0$$

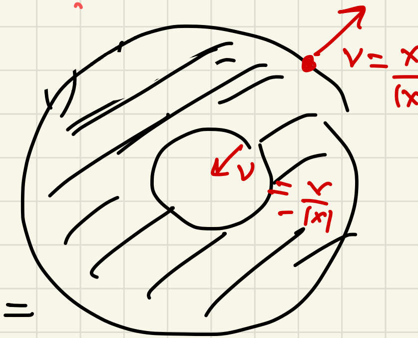
$$\Delta(\lg|x|) = \operatorname{div}(D\lg|x|) = \operatorname{div}\left(\frac{x}{|x|^2}\right) = \frac{2}{|x|^2} - \frac{2}{|x|^3}\sum_i \frac{x_i x_i}{|x|} = 0$$

• fix $\phi \in C_c^\infty(\mathbb{R}^n)$ supp $\phi \subseteq B(0,R)$



$$T_f(\Delta\phi) = \int_{B(0,R)} \frac{c_n}{|x|^{n-2}}\Delta\phi\,dx = \lim_{\varepsilon\to 0^+}\int_{B(0,R)\setminus B(0,\varepsilon)} \frac{c_n}{|x|^{n-2}}\Delta\phi\,dx =$$

= apply divergence theorem to $\operatorname{div}(D\phi \cdot f - Df \cdot \phi) = \lim_{\varepsilon\to 0^+} \cancel{\int_{B(0,R)\setminus B(0,\varepsilon)} c_n \Delta\left(\frac{1}{|x|^{n-2}}\right)\phi\,dx} = 0 \; +$



$$+\lim_{\varepsilon\to 0^+}\int_{\partial B(0,R)} \underbrace{\cancel{D\phi\cdot\left(\frac{x}{|x|}\right)\frac{c_n}{|x|^{n-2}} - \phi\, c_n \frac{x}{|x|^n}\cdot\frac{x}{|x|}}}_{0 \text{ since } \operatorname{supp}\phi \subseteq B(0,R)}\, dS$$

$$+\lim_{\varepsilon\to 0^+}\int_{\partial B(0,\varepsilon)} D\phi\left(-\frac{x}{|x|}\right)\frac{c_n}{|x|^{n-2}} - (2-n)\phi\, c_n \frac{x}{|x|^n}\cdot\left(-\frac{x}{|x|}\right) dS =$$

RECALL $|x| = \varepsilon$

$$= \lim_{\varepsilon\to 0^+}\int_{\partial B(0,\varepsilon)} \underbrace{-\frac{(D\phi(x)\cdot x)\, c_n}{\varepsilon^{n-1}}}_{\to 0} + (2-n)\phi(x)\, c_n \frac{\varepsilon^2}{\varepsilon^n\,\varepsilon}\, dS =$$

$$\int_{\partial B(0,\varepsilon)} \left|-\frac{D\phi(x)\cdot x\; c_n}{\varepsilon^{n-1}}\right| \le \int_{\partial B(0,\varepsilon)} \frac{\|D\phi\|_\infty |x|\, c_n}{\varepsilon^{n-1}} = \int_{\partial B(0,\varepsilon)} \frac{\|D\phi\|_\infty\, \varepsilon\, c_n}{\varepsilon^{n-1}} = \underbrace{\|D\phi\|_\infty c_n\, \varepsilon \; n\omega_n}_{\to 0}$$

$$= \lim_{\varepsilon\to 0}\int_{\partial B(0,\varepsilon)} (2-n)\phi(x)\frac{c_n}{\varepsilon^{n-1}}\, dS = \lim_{\varepsilon\to 0^+}\int_{\partial B(0,\varepsilon)} (2-n)\phi(0)\frac{c_n}{\varepsilon^{n-1}}\, dS + \underbrace{(2-n)\frac{\phi(x)-\phi(0)}{\varepsilon^{n-1}}\, c_n\, dS}_{\le \int_{\partial B} \frac{\|D\phi\|_\infty\, \varepsilon}{\varepsilon^{n-1}} \to 0}$$

$$= \frac{c_n}{\varepsilon^{n-1}}(2-n)\cdot n\omega_n\varepsilon^{n-1}\,\phi(0) = \phi(0). \quad \square$$

Same argument for $n=2$

$$-\Delta T_f = \delta_0 \qquad f = -\frac{1}{2\pi}\log|x|.$$

<u>Ex</u> $\mathbb{R}^{n+1} = \{(x,t) \quad x\in\mathbb{R}^n, \ t\in\mathbb{R}\}$

$$\mathcal{J}u = -\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u + \frac{\partial}{\partial t}u = \frac{\partial u}{\partial t} - \Delta_x u$$

$$f(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}\chi_{(0,+\infty)}(t)$$

$f\in L^1_{loc}(\mathbb{R}^{n+1})$ and $\frac{\partial}{\partial t}T_f - \Delta_x T_f = \delta_0$

T_f is a fundamental solution of $\frac{\partial}{\partial t} - \Delta_x$ ..

Yesterday we saw some examples of functions in $L^1_{loc}(U)$ such that f does not admit weak derivatives, whereas f has ALWAYS derivatives in the sense of distributions! ($D^\alpha T_f$)

In particular f the Cantor Vitali function

which is <u>continuous</u>, <u>constant on $\mathbb{R} \setminus C$</u> where C is the Cantor set (so constant on every interval in $\mathbb{R} \setminus C$), and also monotone increasing by construction.

$T_f \in \mathcal{D}'(\mathbb{R})$ $\quad (T_f)' \neq 0$ but $f' = 0$ a.e.

Observation If f is a monotone non decreasing function on $\mathbb{R}$, then $f \in L^1_{loc}(\mathbb{R})$ (actually it has at most a countable family of jumps discontinuities)

then $(T_f)'$ is a positive distribution that is $(T_f)'(\varphi) \geq 0$ $\forall \phi \geq 0$ $\phi \in C_c^\infty(\mathbb{R})$.

proof $(T_f)'(\varphi) = -\int_{\mathbb{R}} f(x)\,\phi'(x)\,dx$

$\left(\phi'(x) = \lim_{h \to 0^+} \frac{\phi(x+h) - \phi(x)}{h}\right.$ limit is uniform since $\phi \in C_c^\infty(\mathbb{R})$

$= -\lim_{h \to 0^+} \int_{\mathbb{R}} f(x) \frac{\phi(x+h) - \phi(x)}{h}\,dx$

$$= -\lim_{h\to 0^+} \frac{1}{h}\Big[\int_{\mathbb{R}} \underbrace{f(x)\,\phi(x+h)}_{\text{change variable}}\,dx - \int_{\mathbb{R}} f(x)\phi(x)\,dx\Big] =$$

$$= -\lim_{h\to 0^+} \frac{1}{h}\int_{\mathbb{R}} f(x-h)\,\phi(x)\,dx - \int_{\mathbb{R}} f(x)\phi(x)\,dx =$$

$$= -\lim_{h\to 0^+} \frac{1}{h}\int_{\mathbb{R}} \big[f(x-h) - f(x)\big]\,\phi(x)\,dx \geq 0$$

f is non decreasing $f(x-h) - f(x) \leq 0$ since $h > 0$

$\phi(x) \geq 0$ by assumption

$(T_f)'$ is positive distribution $\Longrightarrow$ HAS ORDER 0!

<u>ex</u>: every positive distribution is of order 0

— see ex on moodle page

General fact if $T \in \mathcal{D}'(U)$ is a distrib. of order $r \geq 0$. then it can be extended to a linear functional

$$T : \underbrace{C^r_c(U)}_{\text{functions in } \mathcal{C}^k(U) \text{ with compact support inside } U.} \longrightarrow \mathbb{R}$$

such that $\forall\, K \subset\subset U \quad \exists\, C_K > 0$

$$|T(\phi)| \leq C_K \sum_{|\alpha| \leq r} \|D^\alpha \phi\|_\infty \qquad \forall \phi \in \mathcal{C}^r_c(U), \ \operatorname{supp} \phi \subseteq K$$

$\phi \in \mathcal{C}^r_c(U)$ $\quad \phi_n \in \mathcal{C}^\infty_c(U)$ $\quad \operatorname{supp} \phi_n, \operatorname{supp} \phi \subseteq K$

$\phi_n \to \phi$ in $\mathcal{C}^r(K)$. $\quad T(\phi) = \lim_n T(\phi_n)$

$f \geq 0$ $\Rightarrow (T_f)'$ is a positive distribution of order 0

$\Rightarrow (T_f)'$ is a positive linear functional on $C_c(\mathbb{R})$

$\Rightarrow$ Riesz theorem $\Rightarrow \underbrace{(T_f')(\phi) = \int_{\mathbb{R}} \phi \, d\mu_f}$ $\forall \phi \in C_c(\mathbb{R})$

where μ is a Radon positive measure.

$f : \mathbb{R} \to \mathbb{R}$ monotone NON DECREASING then its derivative in the sense of distribution is a (positive) Radon measure μ_f

$$\forall \phi \in C_c^\infty(\mathbb{R}) \qquad \int_{\mathbb{R}} -\phi' f(x) \, dx = \int_{\mathbb{R}} \phi(x) \, d\mu_f$$

$\mu_f = \mu_f^{ac} + \mu_{ac}^{s}$ $\quad \mu_f^{ac} \ll \mathcal{L}$ $\quad \mu_{ac}^{s} \perp \mathcal{L}$ $\quad \mu_f^{ac}$ has density f'

Going back to the Cantor Vitali function

f is non decreasing, continuous, $f' = 0$ a.e (in $\mathbb{R} \setminus C$)

$\Rightarrow (T_f)' = T_{\mu_f}$ where μ_f is a Radon measure which is SINGULAR with respect to Lebesgue (its abs. continuous part has density $f' = 0$).

It is possible to show (using Riemann - Stieltjes integ.) that $\mu_f(a,b] = \bar{f}(b) - \bar{f}(a)$ (where $\bar{f}$ is a representative of f which is RIGHT CONTINUOUS).

All Radon measure on $\mathbb{R}$ are defined like this ...