Def of derivative of a distribution

$\alpha = (\alpha_1 \dots \alpha_n) \in \mathbb{N}^n \qquad |\alpha| = \alpha_1 + \dots + \alpha_n$

$T \in \mathcal{D}'(U) \qquad U \subseteq \mathbb{R}^n$ open set

$$D^\alpha T(\varphi) := (-1)^{|\alpha|} T(D^\alpha \phi) \quad \forall \phi \in C_c^\infty(U)$$

(so every distribution admits derivatives of every order).

Def of WEAK DERIVATIVE

Let $f \in L^1_{loc}(U)$ and $T_f(\phi) = \int_U f \phi \, dx \quad \forall \phi \in C_c^\infty(U)$

Let $\alpha \in \mathbb{N}^n$. If $\exists\, v_\alpha \in L^1_{loc}(U)$ such that

$D^\alpha T_f = T_{v_\alpha}$ in the sense of distrib.

(that is $D^\alpha T_f(\phi) = T_{v_\alpha}(\phi) \quad \forall \phi \in C_c^\infty(U)$

that is ...

$$(-1)^{|\alpha|}\int_U f\, D^\alpha \phi \, dx = \int_U v_\alpha \cdot \phi \, dx \qquad \forall \phi \in C_c^\infty(U)$$

there v_α is the weak α-derivative of f

($D^\alpha f = v_\alpha$ in the weak sense).

<u>Obs</u> If f admits a weak α-derivative, it is UNIQUE.

if not v_α, w_α such that $\int_U v_\alpha \phi \, dx = \int_U w_\alpha \phi \, dx \ \forall \phi \in C_c^\infty(U)$

$\Rightarrow v_\alpha = w_\alpha$ a.e by the fundam. lemma of the calc. of variations.

In particular $\frac{\partial}{\partial x_i} f$ in the weak sense is the $L^1_{loc}(U)$ function v_i: such that

$$\int_U f \frac{\partial}{\partial x_i} \phi \, dx = - \int_U v_i \phi \, dx \qquad \forall \phi \in C_c^\infty(U)$$

(the integration by parts formula holds (divergence theorem))

Recall that ϕ has compact support inside U!
so $\phi = 0$ on ∂U!

Obs It is not always true that $f \in L^1_{loc}(U)$ admits weak α derivatives for some α.

Ex 1 $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ $H \in L^1_{loc}(\mathbb{R})$

$$(T_H)'(\varphi) = -\int_0^{+\infty} \phi'(x)\,dx = +\phi(0) = \delta_0$$

$T_H' = \delta_0$ (so H has not a weak derivative! $\nexists\, v \in L^1_{loc}$ such that $T_H' = T_v$!

$T_H''(\varphi) = -\phi'(0) \ldots$

Ex 2 take $f \in C^1(\mathbb{R} \setminus \{x_1, \ldots, x_m\})$.

$x_1 < x_2 < \ldots < x_m$

$f \in C^1(-\infty, x_1)$, $C^1(x_1, x_2)$, $C^1(x_2, x_3) \ldots$, $C^1(x_m, +\infty)$

$f' \in L^1_{loc}$ and $\forall i = 1 \ldots m$ $\exists \lim_{x \to x_i^+} f = f(x_i^+)$ $\lim_{x \to x_i^-} f = f(x_i^-)$

CLASSICAL DERIVATIVE

then $\underbrace{(T_f)'(\phi)} = \int_{\mathbb{R}} \phi(x) f'(x)\,dx + \sum_{i=1}^{m} \phi(x_i)\left[f(x_i^+) - f(x_i^-)\right]$

$T_f' \neq T_{f'}$!! even if f' is defined a.e!

$T_f' = T_\mu$ where μ is a signed measure with

$$\mu = \mu_0 + \mu_1$$

$\mu_0 \ll \mathcal{L}$ and ~~μ_0~~ has density $f'(x)$

$\mu_1 = \sum_{i=1}^{m} \alpha_i \, \delta_{x_i}$ $\qquad \alpha_i = f(x_i^+) - f(x_i^-)$

Observation Let $I \subseteq \mathbb{R}$ open interval.

Let $T \in \mathcal{D}'(I)$.

If $T' = 0$ (that means $T'(\phi) = 0 \ \forall \phi \in C_c^\infty(I)$
$\Rightarrow$ so $T(\phi') = 0 \ \forall \phi \in C_c^\infty(I)$)

then T is constant, that is $\exists\, c \in \mathbb{R}$ such that $T = T_c$ (T is the distribution associated to the constant function)

that is $T(\phi) = c \int_I \phi(x)\,dx \quad \forall \phi \in C_c^\infty(I)$.

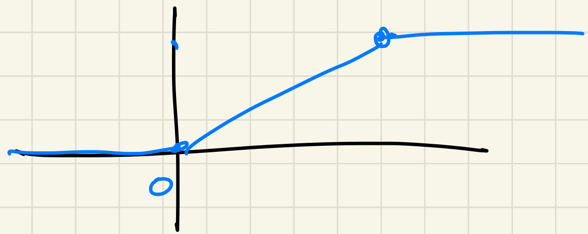
(see moodle page - argue as in the corollary of the fundam. lemma of the calc. of variations)

Ex 3 f continuous in $\mathbb{R}$ but without weak derivative.

$C =$ Cantor set $= \bigcap C_k$

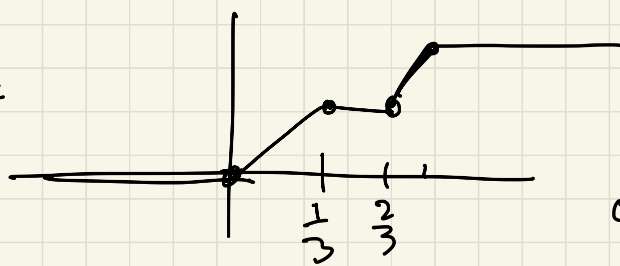
$C_0 = [0\ 1]$ $\qquad C_{k+1} = \frac{1}{3} C_k \cup \left(\frac{1}{3} C_k + \frac{2}{3} \right)$

$f_0(x) =$



$$f_0(x) = \begin{cases} 0 & x \le 0 \\ x & x \in [0,1] \\ 1 & x \geq 1 \end{cases}$$

$f_1(x) =$

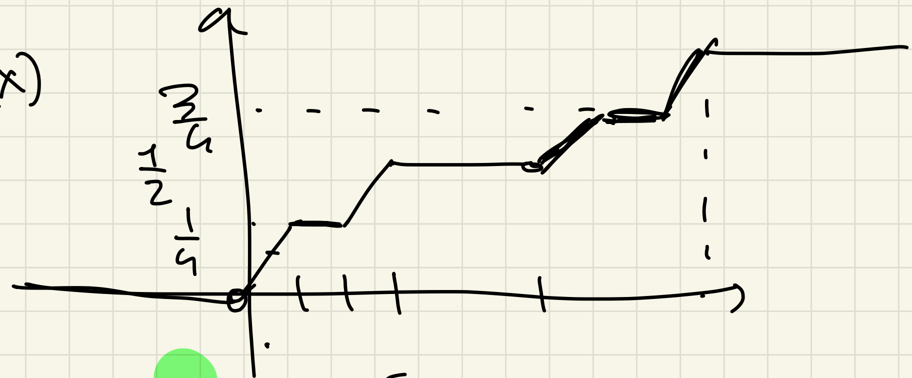


$f_1(x) = f_0(x) \ \forall x \in \mathbb{R} \setminus C_0$

$f_1(x) = \frac{1}{2} \qquad x \in \left(\frac{1}{3}, \frac{2}{3} \right)$

and then affine

$f_2(x)$



$f_2(x)=f_1(x)$
$x\in\mathbb{R}\setminus C_1$

$f_2(x)=\left(\frac{1}{2}\right)^2$ on $(C_1\setminus C_2)\cap\left[0,\frac{1}{2}\right]$

$=\frac{1}{2}+\left(\frac{1}{2}\right)^2$ on $(C_1\setminus C_2)\cap\left[\frac{1}{2},1\right]$

then affine and continuous

and so on

$\forall n\geq k$
$\boxed{f_n(x)=f_k(x) \text{ and constant on } \mathbb{R}\setminus C_k}$

$\|f_{n+1}-f_n\|_\infty\leq \frac{1}{3}\left(\frac{1}{2}\right)^{n+1}$ f_n is Cauchy in $\mathcal{C}([0,1],\|\cdot\|_\infty)$

$f_n \to f$ uniformly on $[0,1]$ (f is the CANTOR VITALI function, its graph is the DEVIL'S STAIRCASE)

f is continuous $f : \mathbb{R} \to [0,1]$

f is constant on every interval contained in $\mathbb{R} \setminus C$ C cantor set.

$\downarrow$

$f' = 0$ on every interval contained in $\mathbb{R} \setminus C$.

$\mathbb{R} \setminus C =$ union of intervals! $|C| = 0$

so $f' = 0$ a.e.

$\underbrace{(T_f)' \neq 0}$ cannot be 0 (if it were $0 \Rightarrow T_f = T_c \Rightarrow f = \text{constant}$ NOT TRUE

f has not weak derivative, $(T_f)'$ is a distribution associated to a measure μ with is SINGULAR with respect to Lebesgue (but no Dirac delta)

Ex 4 $T_{\log|x|}$ $\quad \log|x| \in L^1_{loc}(\mathbb{R})$

$$\left(T_{\log|x|}\right)' = \text{p.v.}\ \frac{1}{x}$$

(so $\log|x|$ has no weak derivative)

$$T'_{\log|x|}(\phi) = -\int_{\mathbb{R}} \log|x|\, \phi'(x)\, dx =$$

take a such that $\operatorname{supp} \phi \subseteq (-a, a)$

$$= -\int_{-a}^{a} \log|x|\, \phi'(x)\, dx = \lim_{\varepsilon \to 0^+} -\int_{-a}^{-\varepsilon} \log|x|\, \phi'(x)\, dx - \int_{\varepsilon}^{a} \log|x|\, \phi'(x)\, dx$$

$$= (\text{by parts}) = \lim_{\varepsilon \to 0} \left[-\log|x|\,\phi(x)\right]_{-a}^{-\varepsilon} + \int_{-a}^{-\varepsilon} \frac{\phi(x)}{x}\, dx + \left[-\log|x|\,\phi(x)\right]_{\varepsilon}^{a} + \int_{\varepsilon}^{a} \frac{\phi(x)}{x}\, dx =$$

$$= \lim_{\varepsilon \to 0^+} \int_{(-a,a)\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x)}{x}\,dx + \underbrace{[\phi(\varepsilon) - \phi(-\varepsilon)] \log \varepsilon}_{\downarrow \; 0 \text{ as } \varepsilon \to 0}$$

(recall $\phi(a) = 0 = \phi(-a)$)

since $\phi(\varepsilon) = \phi(0) + \phi'(0)\varepsilon + o(\varepsilon)$

$\varphi(-\varepsilon) = \phi(0) - \phi'(0)\varepsilon + o(\varepsilon)$

$$[\phi(\varepsilon) - \phi(-\varepsilon)] \log \varepsilon = 2\phi'(0)\,\varepsilon \log \varepsilon + o(\varepsilon)\log \varepsilon$$

$\longrightarrow 0$ as $\varepsilon \to 0^+$.

Ex Let E be a bounded open set of class C^1 in $\mathbb{R}^n$

$$\chi_E(x) = \begin{cases} 0 & x \notin E \\ 1 & x \in E \end{cases} \qquad \chi_E \in L^1(\mathbb{R}^n)$$

$$T_E(\varphi) := \int_E \phi(x)\,dx = T_{\chi_E}(\phi) \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

$$\partial_{x_i} T_E = -\int_E \frac{\partial}{\partial x_i}\phi \, dx = \text{(divergence theorem)} =$$

$$= -\int_{\partial E} \phi(x) \underbrace{\nu_i(x)}_{\downarrow} \, dS(x) = -\int_{\partial E} \phi(x)\nu_i(x) \, d\mathcal{H}^{n-1}(x).$$

$\nu(x)$ exterior normal to E at x

$\nu_i(x) =$ i-component

∇T_E is a vector valued distribution $(\nabla T_E : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}^n)$ supported on ∂E and with density $-\nu_E(x)$ (with respect to $\mathcal{H}^{n-1}$ - measure)

__Def__ $\mathcal{L}$ is a LINEAR DIFFERENTIAL OPERATOR of order k ($\in \mathbb{N}$) with CONSTANT COEFFICIENTS if $\exists\, c_\alpha \in \mathbb{R}$ $\forall |\alpha| \le k$, $\forall u \in C^k(\mathbb{R}^n)$

$$\mathcal{L}(u) := \sum_{|\alpha|\le k} c_\alpha D^\alpha u \in C^0(\mathbb{R}^n)$$

ex $\mathcal{F}$ of order 1:

ex $\mathcal{F}(u) = c_0 u + \sum_i c_i \frac{\partial}{\partial x_i} u$ $\qquad \mathcal{F} : C^1(\mathbb{R}^n) \to C^0(\mathbb{R}^n)$

$\mathcal{F}$ of order 2

ex $\mathcal{F}(u) = \Delta u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u$

ex. $\mathcal{F}(u) = \mathrm{tr}(A_0 \cdot D^2 u) +$ $\qquad A_0 \in M^{n \times n}(\mathbb{R})$

$+ \sum_i c_i \frac{\partial}{\partial x_i} u + c_0 u$

Let $T \in \mathcal{D}'(U)$ $\mathcal{J}$ be a linear diff. operator of order k with constant coefficients:

$\mathcal{J}(u) = \sum_{|\alpha| \le k} c_\alpha D^\alpha u$

$\mathcal{J}(T)$ is the distribution s.t. that

$$\mathcal{J}(T)(\phi) := \sum_{|\alpha| \le k} c_\alpha D^\alpha T(\phi) = \sum_{|\alpha| \le k} c_\alpha (-1)^{|\alpha|} T(D^\alpha \phi)$$

$\forall \phi \in C_c^\infty(U)$

$$\underset{\text{by linearity}}{=} T\Big(\sum_{|\alpha| \le k} c_\alpha (-1)^{|\alpha|} D^\alpha \phi \Big)$$

ex
$$\Delta T(\varphi) := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} T(\phi) = \sum_{i=1}^{n} (-1)^2 T\Big(\frac{\partial^2}{\partial x_i^2} \phi\Big) =$$

$$\underset{\text{by linearity}}{=} T\Big(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \phi \Big) = T(\Delta \phi).$$

Def Let $\mathcal{J}$ be a linear differential operator with constant coefficients

$T \in \mathcal{D}'(\mathbb{R}^n)$ is a FUNDAMENTAL SOLUTION of T if

$$\mathcal{J}(T) = \delta_0$$

(that is $\mathcal{J}(T)(\varphi) = \varphi(0) \ \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$)