T distrib

$T : \underbrace{\mathcal{C}^\infty_c(U)} \to \mathbb{R}$ linear and (seq) continuous

T continuous means that $\forall\, \phi_n \to \phi$ in $\mathcal{C}^\infty_c(U)$

it holds $T(\varphi_n) \to T(\varphi)$

$\updownarrow$ equivalent

$\forall K \subset\subset U \quad \exists\, C_K > 0 \quad p_K \in \mathbb{N}$ such that

$\circledast \quad |T(\varphi)| \leq C_K \cdot \sum_{|\alpha| \leq p_K} \|D^\alpha \phi\|_\infty \qquad \forall \phi \in \mathcal{C}^\infty_c(U)$

supp $\phi \subseteq K$.

$\forall K \subset\subset U$ we consider the MINIMUM p_K for which

$\circledast$ holds

$p :=$ order of the distribution $= \sup_{K \subset\subset U} p_K$

Ex $\quad f \in L^1_{loc}(U) \qquad T_f(\phi) := \int_U \phi \cdot f \, dx$

it has order 0.

supp $\phi \subseteq K \subset\subset U$

$$|T_f(\phi)| = \left| \int_K \phi(x) \cdot f(x) \, dx \right| \leq \|\phi\|_\infty \cdot \underbrace{\|f\|_{L^1(K)}}_{\int_K |f(x)| \, dx}$$

μ is a Radon measure $\qquad T_\mu(\phi) := \int_U \phi \, d\mu$

it has order 0.

supp $\phi \subseteq K \subset\subset U \qquad |T_\mu(\phi)| = \left| \int_K \phi \, d\mu \right| \leq \|\phi\|_\infty \, \mu(K)$

we fix $i \in \{1 \dots n\}$,

$$T(\varphi) = \underbrace{\phi_{x_i}(0)}_{\downarrow} \qquad T \in \mathcal{D}'(\mathbb{R}^n)$$

$$T : \mathcal{C}^\infty_c(\mathbb{R}^n) \longrightarrow \mathbb{R}$$

$$= D^\alpha \phi(0) \qquad \alpha = (0 \dots, \underset{\substack{\downarrow \\ i\text{-th}}}{1}, 0 \dots 0)$$

obs this distribution has order 1.

$$|T(\varphi)| \leq 1 \, \|\phi_{x_i}\|_\infty \leq \sum_{|\alpha| \leq 1} \|D^\alpha \phi\|_\infty$$

the order is ≤ 1

$$\phi \in \mathcal{C}^\infty_c(U) \qquad \phi_{x_i}(0) = 1 \qquad \phi^k(x) = \phi(kx)$$

$$\phi^k_{x_i}(0) = k \qquad k \in \mathbb{N}$$

$$T(\phi^k) = k \qquad \forall\, k \in \mathbb{N}$$

Take C compact in $\mathbb{R}^n$
such that $\operatorname{supp} \phi^k \subseteq C$

if T have order 0 then

$$|T(\psi)| \leq C_C \, \|\psi\|_\infty \qquad \forall\, \psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$$
$$\operatorname{supp} \psi \subseteq C$$

$$k = |T(\phi^k)| \leq C_C \|\phi\|_\infty$$

So the order of T cannot be 0.

Another important example of distn of order 1 (in DIMENSION 1), in $\mathcal{D}'(\mathbb{R})$

p.v. $\frac{1}{x}$ (principal value of $\frac{1}{x}$)

$\forall \phi \in C_c^\infty(\mathbb{R})$

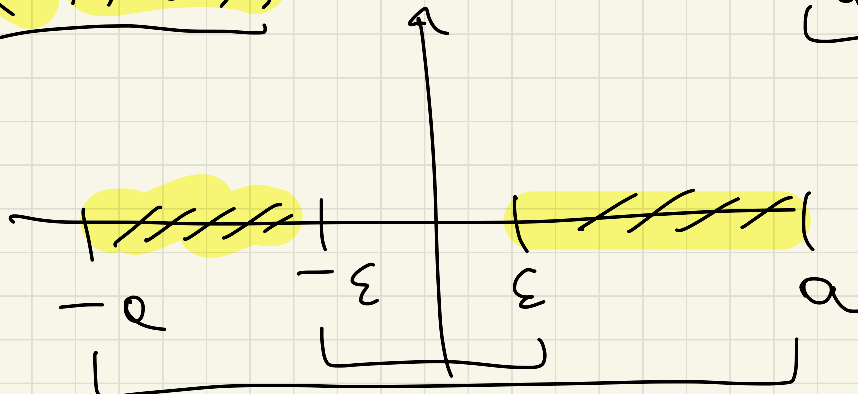
$$T(\phi) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{\phi(x)}{x}\, dx$$

fix ϕ, fix $(-a, a)$ such that supp $\phi \subseteq (-a, a)$

$$= \lim_{\varepsilon \to 0^+} \int_{(-a,a) \setminus (-\varepsilon, \varepsilon)} \frac{\phi(x)}{x}\, dx = \lim_{\varepsilon \to 0^+} \int_{(-a,a) \setminus (-\varepsilon, \varepsilon)} \frac{\phi(x) - \phi(0)}{x}\, dx$$

$$\int_{(-a,a)\setminus(-\varepsilon,\varepsilon)} \frac{\phi(0)}{x}\cdot dx = 0 = \int_{-a}^{-\varepsilon} \frac{\phi(0)}{x}\,dx + \int_{\varepsilon}^{a} \frac{\phi(0)}{x}\,dx$$

$$= 0$$



fix $\varepsilon > 0$

$0 < \varepsilon < a$

$$\int_{(-a,a)\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x)}{x}\,dx = \int_{(-a,a)\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x)-\phi(0)}{x}\,dx$$

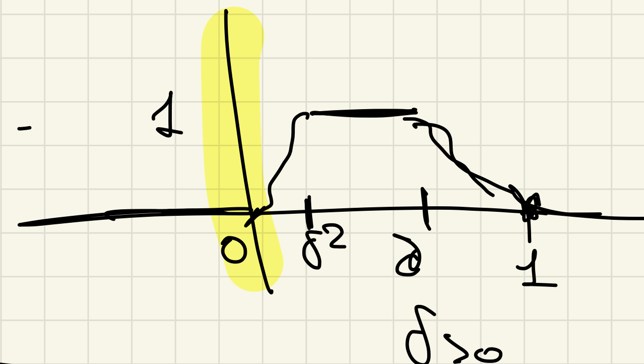
$$\phi(x)-\phi(0) = \int_0^x \phi'(z)\,dz = \int_0^1 \phi'(tx)\,x\,dt$$

$z = tx$

$$= \underbrace{\int_{(-a,a)\setminus(-\varepsilon,\varepsilon)} \int_0^1 \phi'(tx)\,dt\,dx}\; = \int_{(-a,a)\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x)}{x}\,dx$$

(so the limit exists as $\varepsilon \to 0^+$

This p.v. $\frac{1}{x}$ has order 1.



generalization to $\mathbb{R}^n$

$$\frac{1}{x} \rightsquigarrow f(x) \qquad f(rx) = \frac{f(x)}{r^n} \quad r>0$$

p.v. $f(x)$ $\qquad \int_{\partial B(0,1)} f(s)\,dS = 0$

Support of a distribution $\quad T \in \mathcal{D}'(U)$

$\operatorname{supp} T = U \setminus \omega_0 \quad$ (relatively closed inside U)

$\omega_0 =$ union of open sets $V \subseteq U$ such that $T(\varphi) = 0 \quad \forall \phi \in C_c^\infty(U)$

$\operatorname{supp} \phi \subseteq V$

ex $\quad T = \delta_0 \in \mathcal{D}'(\mathbb{R}^n) \qquad \delta_0(\phi) = \phi(0)$

$\operatorname{supp} T = \{0\} \quad$ (if $\operatorname{supp} \phi \subseteq \mathbb{R}^n \setminus \{0\}$

$T(\phi) = 0$

if $T = T_f$ for $f \in L^1_{loc}(U)$

$\operatorname{supp} T_f = \operatorname{supp} f$

if $T = T_\mu$ for μ Radon meas.

$\operatorname{supp} T_\mu = \operatorname{supp} \mu$

<u>Obs</u> If $\phi \in C^\infty_c(U)$ is such that

$\phi(x) = 0 \ \forall x \in \operatorname{supp} T \subset U$

$\not\Rightarrow T(\varphi) = 0$

ex. $T(\phi) = \phi_{x_i}(0)$ $\quad T \in \mathcal{D}'(\mathbb{R}^n)$

$\boxed{1)\ \operatorname{supp} T = \{0\}}$

$\operatorname{supp} \phi$ is a compact set inside $\mathbb{R}^n \setminus \{0\} \Longrightarrow \phi_{x_i}(0) = 0$

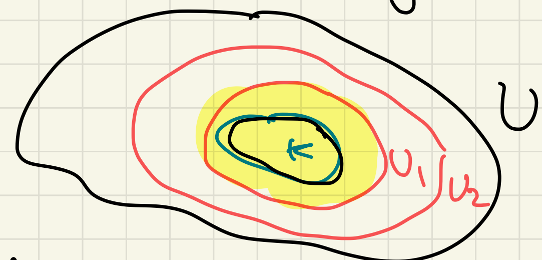
$T(\phi) = 0$.

2 if $\phi(0) = 0 \not\Longrightarrow T(\phi) = \phi_{x_i}(0) \neq 0$

non true in general.

Observation

Let $T \in \mathcal{D}'(U)$ if supp T is a compact set inside U then the order of T is finite.

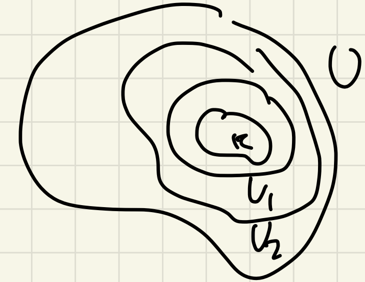


proof Fix $K = \text{supp } T$

fix U_1 open bounded set $U_1 \subsetneq U$ such that $K \subseteq U_1$

fix U_2 open bdd set $U_1 \subseteq U_2 \subsetneq U$

$\overline{U_1}, \overline{U_2}$ are compact sets

$\chi \in \mathcal{C}^\infty_c(U)$ such that 

$$\chi(x) \equiv 1 \quad \forall x \in \overline{U_1}$$

$$\chi(x) \equiv 0 \quad \text{in} \quad x \in U \setminus U_2$$

$$0 \le \chi \le 1$$

$\phi \in \mathcal{C}^\infty_c(U)$ $\qquad \phi \cdot (1-\chi) \in \mathcal{C}^\infty_c(U)$

$$\underbrace{\phi(1-\chi) \equiv 0 \quad \text{on} \quad \overline{U_1}}$$

$T(\phi(1-\chi)) = 0$ $\qquad T(\varphi) = T(\phi \cdot \chi)$

$\forall \phi \in \mathcal{C}_c^\infty(U)$

$T(\varphi) = T(\varphi\chi)$

$\phi\chi \in \mathcal{C}_c^\infty(U)$ $\quad \operatorname{supp}(\phi\chi) \subseteq \underbrace{\overline{U_2} \text{ compact}}$

$$|T(\varphi)| = \underbrace{|T(\varphi\chi)| \leq C_{\overline{U_2}} \cdot \sum_{|\alpha| \leq p_{\overline{U_2}}} \|D^\alpha(\chi\phi)\|_\infty} \leq$$

$$\underbrace{\leq \boxed{C_{\overline{U_2},\chi}} \sum_{|\alpha| \leq p_{\overline{U_2}}} \|D^\alpha \phi\|_\infty}$$

$$|D^\alpha(\phi\chi)| = \Big|\sum_{\beta} \sum_{|\beta| \leq \alpha} D^\beta \phi \cdot D^{\alpha-\beta} \chi\Big|$$

the order is $p_{\overline{U_2}}$

"Derivative of a distribution". $U \subseteq \mathbb{R}^n$.

$T \in \mathcal{D}'(U)$ $\qquad \alpha = (\alpha_1 \dots \alpha_n)$ $\qquad \alpha_i \in \mathbb{N}$

$D^\alpha T \in \mathcal{D}'(U)$ $\qquad \boxed{D^\alpha T(\phi) := (-1)^{|\alpha|} T(D^\alpha \phi)}$

$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$

$$\underbrace{D^\alpha \phi} = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \cdots \frac{\partial^{\alpha_n}}{\partial x_n} \phi$$

$f \in \mathcal{C}^1(U)$ $\qquad \alpha = (1, 0 \dots 0)$

$$(D^\alpha T_f)(\phi) = (-1) \cdot T_f \cdot (\partial x_1 \phi) = T_{\partial x_1 f}(\phi)$$

if f, ϕ are smooth enough $\phi \in C_c^\infty(U)$.

$$\int_U D^\alpha f \cdot \phi \, dx = (-1)^{|\alpha|} \int_U f \cdot D^\alpha \phi \, dx$$

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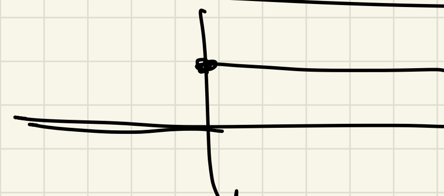
ex $\mathcal{D}'(\mathbb{R})$



$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$H(x) \in L^1_{loc}(\mathbb{R})$

$\forall \phi \in C_c^\infty(\mathbb{R})$

$$T_H(\phi) = \int_0^{+\infty} \phi(x)\, dx = \underbrace{\int_{\mathbb{R}} \phi(x) H(x)\, dx}$$

$$(T_H)'(\phi) \underset{\uparrow \text{ by def.}}{=} (-1)^1 \underbrace{T_H(\phi')}_{} =$$

$$\underbrace{D^\alpha T_H}_{\alpha=1}(\phi)$$

$$\phi \in C_c^\infty(\mathbb{R})$$

$$= (-1) \cdot \int_0^{+\infty} \phi'(x)\,dx =$$

$$= (-1) \cdot (-\phi(0)) = \phi(0)$$

$$(T_H)' = \delta_0 \,.$$

If $f \in L^1_{loc}(U)$ $\qquad T_f(\phi) = \int_U \phi f \, dx$

$\phi \in C_c^\infty(U)$

$\forall i = 1 \dots n \qquad \alpha = (0 \dots \underset{i}{1} \dots 0) \qquad |\alpha| = 1$

$$\left(\frac{\partial}{\partial x_i} T_f\right)(\phi) = \underbrace{(-1) \cdot T_f\left(\frac{\partial}{\partial x_i}\phi\right)}$$

if for every i $\exists$ $v_i \in L^1_{loc}(U)$ such that

$$T_f\left(\frac{\partial}{\partial x_i}\phi\right) := -\int_U \phi(x)\, v_i(x)\, dx \quad \forall \phi \in C_c^\infty(U)$$

$\left(\frac{\partial}{\partial x_i} T_f \text{ "=" } T_{v_i}\right)$ in the sense of distribution then $v_i = \frac{\partial}{\partial x_i} f$ in the weak sense

(v_i is the weak derivative of f)

$$\underbrace{\left(\frac{\partial}{\partial x_i} T_f\right)(\phi)}_{\parallel} = T_{v_i}(\phi) \quad \forall \phi \in C_c^\infty(U)$$

$$-\int_U f(x)\, \partial_{x_i}\phi(x)\,dx = +\int_U v_i(x)\,\phi(x)\,dx$$

$$\forall \phi \in C_c^\infty(U)$$

$\alpha \doteq (\alpha_1 \dots \alpha_n)$

$v_\alpha \in L^1_{loc}(U)$ is the α-weak derivative of f if

$\forall \phi \in C_c^\infty(U) \qquad D^\alpha T_f \text{ "=" } D_{v_\alpha}$

$$\int_U v_\alpha(x)\, \phi(x)\, dx = (-1)^{|\alpha|} \int_U f(x)\, D^\alpha \phi(x)\, dx$$