Theory of DISTRIBUTIONS (Schwartz '50)

$U \subseteq \mathbb{R}^n$ open

$$\underbrace{\mathcal{C}^\infty_c(U)}_{\mathcal{D}(U)}$$

$f_n \to f$ in $\mathcal{C}^\infty_c(U)$

if $\exists K \subset\subset U$ $\operatorname{supp} f_n, \operatorname{supp} f \subseteq K$

$D^\alpha f_n \longrightarrow D^\alpha f$ ~~in~~ uniformly

$\| D^\alpha f_n - D^\alpha f \|_\infty \longrightarrow 0$ $\forall$

Def A linear functional $T : \mathcal{C}^\infty_c(U) \longrightarrow \mathbb{R}$ is a DISTRIBUTION if it is (sequentially) continuous w.r.t. the convergence we defined in $\mathcal{C}^\infty_c(U)$. $f_n \to f \Rightarrow T(f_n) \to T(f)$.

$\mathcal{D}'(U)$ = space of all distributions in $C_c^\infty(U)$.

$T_n : C_c^\infty(U) \to \mathbb{R}$

$T : C_c^\infty(U) \to \mathbb{R}$

I put a notion of convergence

$T_n, T \in \mathcal{D}'(U)$ $\qquad T_n \to T$ (IN THE SENSE OF DISTRIBUTION)

if $\forall \phi \in C_c^\infty(U)$ $\quad T_n(\phi) \to T(\varphi)$

<u>Obs 1</u> $\forall f \in L^1_{loc}(U)$ $\quad (f \in L^p_{loc}(U))$

can be associated uniquely to a distribution

$$\forall \phi \in C^\infty_c(U), \quad \boxed{T_f(\phi) = \int_U \phi(x) f(x)\, dx}$$

<u>Obs 2</u> if μ is a Radon measure it is associated uniquely to a dist.

$$\forall \phi \in C^\infty_c(U) \qquad T_\mu(\phi) = \int_U \phi(x)\, \underline{d\mu}$$

$\forall A \subseteq \mathbb{R}^n$

ex $\delta_0(A) = \begin{cases} 0 & 0 \notin A \\ 1 & 0 \in A \end{cases}$ (Dirac-delta measure)

$\downarrow$ is associated in a unique way to the dist$_n$

$\boxed{\text{Assume } 0 \in U}$

$$\delta_0(\phi) = \phi(0) \quad \forall \phi \in C_c^\infty(U)$$

$\delta_0 \in \mathcal{D}'(\mathbb{R}^n)$

- Not every distribution is associated to a $L^1_{loc}(U)$ function!

e.g. $\delta_0(\phi)$ cannot be written as $T_f(\phi)$ for any $f \in L^1_{loc}$.

$\nexists f \in L^1_{loc}$ such that $\forall \phi \in C_c^\infty(U)$ $\phi(0) = \int_U f(x)\phi(x)\,dx$

$$\phi_k(x) = \begin{cases} e^{\frac{1}{k|x|^2-1}} & |x| < \frac{1}{\sqrt{k}} \quad x \in B(0, \frac{1}{\sqrt{k}}) \\ 0 & \end{cases}$$

$$\phi_k \in \mathcal{C}_c^\infty(\mathbb{R}^n)$$

$$\delta_0(\phi_k) = \phi_k(0) = \frac{1}{e} \quad \forall k$$

$$f \in L^1_{loc}(\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} f(x)\phi_k(x)dx = \int_{B(0,1)} f(x)\, e^{\frac{1}{k|x|^2-1}} \chi_{B(0,\frac{1}{\sqrt{n}})}(x)dx \xrightarrow{\text{Lebesgue}} 0$$

Obs Actually some arg.

$f \in L^1(\mathbb{R}^n)$ $\qquad \int_{\mathbb{R}^n} f(x)\,dx = 1$

$$f_t(x) := \frac{1}{t^n} f\left(\frac{x}{t}\right) \qquad t \in \mathbb{R} \quad \underline{\underline{t>0}}$$

$$\boxed{t \to 0^+} \quad T_{f_t} \longrightarrow \delta_0$$

in the sense of distribution.

(we have just to prove that $T_{f_t}(\phi) \to \phi(0)$ $\forall \phi \in C_c^\infty(\mathbb{R}^n)$

$$T_{f_t}(\phi) = \int_{\mathbb{R}^n} \phi(x) \frac{1}{t^n} f\left(\frac{x}{t}\right) dx =$$

$$= \int_{\mathbb{R}^n} \phi(tx)\, f(x)\, dx \xrightarrow{t \to 0} \phi(0)$$

(using the fact that $\int_{\mathbb{R}^n} f(x)\, dx = 1$

$f \in C^1$

An equivalent characterization of distribution

$T : \mathcal{C}^\infty_c(U) \to \mathbb{R}$ linear

it is equivalent 1 | T is sequentially continuous

2 $\forall K \subset\subset U \quad \exists C_K > 0 \quad \exists p_K \in \mathbb{N}$

such that

$$\forall \phi \in \mathcal{C}^\infty_c(U) \quad \operatorname{supp} \phi \subseteq K$$

it holds $\quad |T(\phi)| \leq C_K \cdot \left[\sum_{|\alpha| \leq p_K} \|D^\alpha \phi\|_\infty \right]$

② does not hold $\Rightarrow$ ① does not hold

(① $\Rightarrow$ ②)

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by contradiction

$\exists K \subset\subset U \quad \forall j \in \mathbb{N} \quad \exists \phi_j \in C_c^\infty(U)$

$\operatorname{supp} \phi_j \subseteq K$

$$|T(\phi_j)| \geq j \cdot \sum_{|\alpha| \leq j} \|D^\alpha \phi_j\|_\infty$$

$$\psi_j = \frac{\phi_j}{|T(\phi_j)|} \in C_c^\infty(U) \qquad \operatorname{supp} \psi_j \subseteq K \qquad T(\psi_j) \to 0$$

$$\boxed{|T(\psi_j)| = 1, \quad \sum_{|\alpha| \leq j} \|D^\alpha \psi_j\|_\infty \leq \frac{1}{j}}$$

$|T(\psi_j)| = 1$

(2) $\Rightarrow$ (1) $\qquad \phi_n \to \phi$ in $\mathcal{C}_c^\infty(U)$

$\downarrow$

$\exists K$, $\operatorname{supp} \phi_n, \phi \subseteq K$

$$\| D^\alpha \phi_n - D^\alpha \phi \|_\infty \longrightarrow 0$$

by (2)

$$| T(\phi_n - \phi) | \leq \underbrace{C_K \sum_{|\alpha| \leq p_K} \| D^\alpha \phi_n - D^\alpha \phi \|_\infty}_{\downarrow \atop 0}$$

So $T(\phi_n) \longrightarrow T(\phi)$

Assume that T is a distribution such that

$\forall K \subset\subset U \quad \exists C_K > 0 \quad \exists p \in \mathbb{N}$ (INDEPENDENT on K)

$$|T(\phi)| \leq C_K \sum_{|\alpha| \leq p_K} \|D^\alpha \phi\|_\infty \leq C_K \sum_{|\alpha|\leq p} \|D^\alpha \phi\|_\infty$$

$\forall \phi \in C_c^\infty(U)$

$\operatorname{supp} \phi \subseteq K$

"$p = \sup_{K \subset\subset U} p_K$"

If p is independent on $K \Rightarrow$
p = ORDER of the DISTRIBUTION

p = order of the distribution

$$= \sup_{K \subset\subset U} p_K$$

where p_K is for every $K \subset\subset U$ the <u>minimum</u> natural number such that

$$|T(\varphi)| \leq C_K \sum_{|\alpha| \leq p_K} \|D^\alpha \phi\|_\infty$$

$\forall \phi \in C_c^\infty(U)$ $\operatorname{supp} \phi \subseteq K$.

Obs $f \in L^1_{loc}(U) \longrightarrow T_f$ is of order 0.

$$|T_f(\phi)| = \left| \int_U f(x)\phi(x)\,dx \right| \leq$$

$$\leq \|\phi\|_\infty \cdot \underbrace{\|f\|_{L^1(K)}}_{= C_K}$$

Also if μ is a Radon measure T_μ is of order 0.

Ex $\quad T : \phi \longrightarrow \phi_{x_i}(0) \quad$ i fixed

in $\mathcal{D}'(\mathbb{R}^n)$

T has order 1.

trivial

$\forall K \subset\subset \mathbb{R}^n \quad \exists\, C_K$ such that

$$|T(\phi)| = |\phi_{x_i}(0)| \le \|\phi\|_\infty + \sum_j \|\phi_{x_j}(0)\|_\infty$$

this says that the order $p \le 1$