Functional Analysis, A.A. 2019-2020

Exam- November 27- Time 2 hours

Problem 1.

(1) Give the definition of absolutely continuous measure and of singular measure (with respect to the Lebesgue measure in $\mathbb{R}$).

(2) Consider the measure μ defined in $\mathcal{M}$ (the σ-algebra of Lebesgue measurable sets) as follows: for every $A \subseteq \mathbb{R}$, measurable,

$$\mu(A) = \text{number of elements } z \in \mathbb{Z}, \text{ such that } z \in A.$$

Check that it is a measure, and write if μ is either absolutely continuous with respect to $\mathcal{L}$ (Lebesgue measure) or singular with respect to $\mathcal{L}$ or none of them.

Hint: recall that $\mathcal{L}(\mathbb{Z}) = 0$.

Problem 2.

(1) State the Hölder inequality for $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$.

(2) Let $g \geq 0$ such that $g \in L^1(\mathbb{R})$. Let $f : \mathbb{R} \to \mathbb{R}$ a measurable function. Show that if for some $p > 1$

$$\int_{\mathbb{R}} |f(x)|^p g(x)dx < +\infty \qquad \text{that is } f(x)[g(x)]^{\frac{1}{p}} \in L^p(\mathbb{R})$$

then

$$\int_{\mathbb{R}} |f(x)|g(x)dx < +\infty, \qquad \text{that is } f(x)g(x) \in L^1(\mathbb{R}).$$

Problem 3. Let H be a Hilbert space on $\mathbb{R}$.

(1) Let $V \subset H$. Define the orthogonal subspace $V^{\perp}$.

(2) State the orthogonal projection theorem.

(3) Let $H = L^2(-\pi, \pi)$. Let

$$e_1(x) \equiv \frac{1}{\sqrt{2\pi}}, \qquad e_2(x) = \frac{\sin x}{\sqrt{\pi}}.$$

Check that $\{e_1, e_2\}$ is a orthonormal set in $L^2(-\pi, \pi)$.

Hint: recall that $\int_a^b \sin^2 x dx = \left[\frac{x - \cos x \sin x}{2}\right]_a^b$.

(4) Compute the orthogonal projection of x and of x^2 on the subspace $V \subset L^2(-\pi, \pi)$ which has orthonormal basis $\{e_1, e_2\}$.

Hint: recall that $\int_a^b x \sin x dx = [-x \cos x + \sin x]_a^b$.

Sketch of solutions

Solution 1. (1) $\mu << \mathcal{L}$ (μ is absolutely continuous with respect to Lebesgue) if for any $E \in \mathcal{M}$ such that $\mathcal{L}(E) = 0$ there holds that $\mu(E) = 0$.

$\mu \perp \mathcal{L}$ (μ is singular with respect to Lebesgue) if there exist $A, B \in \mathcal{M}$ such that $\mathbb{R} = A \cup B$, $A \cap B = \emptyset$ and $\mu(A) = 0$, $\mathcal{L}(B) = 0$.

(2) Note that $\mu(\emptyset) = 0$. Moreover, If $(A_i)_i$ is a sequence of pairwise disjoint measurable sets then by definition $\mu(\cup_i A_i)$ =number of $z \in \mathbb{Z}$ such that $z \in \cup_i A_i$. But $z \in \cup_i A_i$ if and only if $z \in A_i$ for exactly one i (since the sets are disjoint). Therefore $\mu(\cup_i A_i) = \sum_i$number of $z \in \mathbb{Z}$ such that $z \in A_i = \sum_i \mu(A_i)$. This implies that μ is a measure.

Note that $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Z}) \cup \mathbb{Z}$ and $\mathcal{L}(\mathbb{Z}) = 0$, whereas $\mu(\mathbb{R} \setminus \mathbb{Z}) = 0$. Therefore $\mu \perp \mathcal{L}$.

Solution 2.

(1) Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $f(x)g(x) \in L^1(\mathbb{R})$ and there holds $\|fg\|_1 \le \|f\|_p \|g\|_q$.

(2) Let $p > 1$ and fix $q = \frac{p}{p-1}$ the conjugate exponent of p (so that $\frac{1}{p} + \frac{1}{q} = 1$). Since $g \ge 0$ and $g \in L^1(\mathbb{R})$, we get that

$$|[g(x)]^{\frac{1}{q}}|^q = g(x) \in L^1(\mathbb{R})$$

and so $[g(x)]^{\frac{1}{q}} \in L^q(\mathbb{R})$. So by Hölder inequality we get

$$f(x)[g(x)]^{\frac{1}{p}}[g(x)]^{\frac{1}{q}} = f(x)g(x) \in L^1(\mathbb{R}).$$

Solution 3.

(1) $V^\perp = \{h \in H, \mid (v, h) = 0 \ \forall v \in V\}$.

(2) Let $V \subseteq H$ be a closed subspace in H. Then for all $h \in H$ there exists a unique element $v \in V$ and a unique element $w \in V^\perp$ such that $h = v + w$. Moreover v is called the orthogonal projection of h in V, since $h - v \in V^\perp$.

(3) It is sufficient to check that $(e_1, e_2) = 0$ and that $\|e_1\|_2 = 1 = \|e_2\|_1$. So,

$$(e_1, e_2) = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{\sin x}{\sqrt{\pi}} dx = \frac{1}{\sqrt{2}\pi} \int_{-\pi}^{\pi} \sin x dx = 0$$

since $\sin x$ is a odd function. Moreover

$$\|e_1\|_2^2 = \int_{-\pi}^{\pi} \frac{1}{2\pi} dx = 1$$

and

$$\|e_2\|_2^2 = \int_{-\pi}^{\pi} \frac{1}{\pi} \sin^2 x dx = \frac{1}{\pi} \left[\frac{x - \cos x \sin x}{2} \right]_{-\pi}^{\pi} = \frac{1}{2\pi}(\pi - 0 - (-\pi - 0)) = \frac{2\pi}{2\pi} = 1.$$

(4) By the theorem on the computation of the orthogonal projection we have that

$$P_V(x) = a_1 e_1(x) + a_2 e_2(x) \qquad P_V(x^2) = c_1 e_1(x) + c_2 e_2(x)$$

where

$$a_1 = (x, e_1) = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} x dx = 0$$

since x is a odd function,

$$\begin{aligned} a_2 = (x, e_2) &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} x \sin x dx = \frac{1}{\sqrt{\pi}} \left[-x\cos x + \sin x\right]_{-\pi}^{\pi} \\ &= \frac{1}{\sqrt{\pi}}(-\pi \cos \pi - (\pi)\cos(-\pi)) = \frac{2\pi}{\sqrt{\pi}} = 2\sqrt{\pi} \\ c_1 = (x^2, e_1) &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} x^2 dx = \frac{1}{\sqrt{2\pi}} \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3\sqrt{2\pi}} \\ c_2 = (x^2, e_2) &= \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} x^2 \sin x dx = 0 \end{aligned}$$

since $x^2 \sin x$ is a odd function. Therefore the orthogonal projections of x and x^2 in V are given by

$$P_V(x) = 0e_1(x) + 2\sqrt{\pi}e_1(x) = 2\sqrt{\pi}\frac{1}{\sqrt{\pi}} \sin x = 2\sin x$$

$$P_V(x^2) = \frac{2\pi^3}{3\sqrt{2\pi}} e_1(x) + 0e_2(x) = \frac{2\pi^3}{3\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} = \frac{\pi^2}{3}.$$