Functional Analysis

Exam- 13 December, 2022- 90 minutes

Exercise 1.

1. Give the definition of absolutely continuous measure and of singular measure (with respect to the Lebesgue measure in $\mathbb{R}$).
2. Recall the characterization (in terms of the density function..) of Borelian σ-finite measures on $\mathbb{R}$ which are absolutely continuous with respect to the Lebesgue measure.
3. Let
$$f(x) = \begin{cases} \frac{1}{3} & -1 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$$
Is f the density of an absolutely continuous measure μ? If the answer is positive, compute $\mu(\mathbb{R})$, $\mu[0,1]$ and $\mu[-2,-1]$.

Exercise 2.

Let $H = M^2(\Omega, \mathbb{P}, \mathcal{F})$ the space of random variables with bounded second moment,

1. State the orhogonal projection theorem on Hilbert spaces.
2. Recall the definition of bounded linear operator $T : H \to H$, where H is a Banach space. Recall the definition of norm of a bounded linear operator.
3. Consider the set
$$C = \{X \in H \ | \mathbb{E}(X) = 0\}.$$
Show that C is a closed subspace of H.
4. Consider the map $T : H \to H$ such that $T(X) = X - \mathbb{E}(X)$. Show that this is a bounded linear operator. Compute the norm of this operator.
5. Show that for every $X \in H$, $X - \mathbb{E}(X)$ is orthogonal to every constants $k \in \mathbb{R}$. Note that $X - \mathbb{E}(X) \in C$. Compute the orthogonal space $C^{\perp}$.
6. Given $X \in H$, find the best constant $c \in \mathbb{R}$ such that
$$\mathbb{E}(X - c)^2 = \min_{k \in \mathbb{R}} \mathbb{E}(X - k)^2.$$

Sketch of solutions

Solution 1.

3 Observe that $f(x) \geq 0$, f is measurable and $\int_{\mathbb{R}} f(x)dx = \int_{-1}^{2} \frac{1}{3}dx = 1$. So $f \in L^1(\mathbb{R})$, which implies that f is the density of a finite Borelian measure μ which is absolutely continuous with respect to the Lebesgue measure. Moreover for every $A \in \mathcal{B}$, $\mu(A) = \int_A f(x)dx = \frac{1}{3}|A \cap (-1, 2)|$. This implies that $\mu[0, 1] = \frac{1}{3}$ and $\mu[-2, -1] = 0$.

Solution 2.

3 Observe that if $X, Y \in C$, then $\alpha X + \beta Y \in C$ for every $\alpha, \beta \in \mathbb{R}$ since $\mathbb{E}(\alpha X + \beta Y) = \alpha\mathbb{E}(X) + \beta\mathbb{E}(Y) = 0$. Moreover if $X_n \in C$ for every n and $X_n \to X$ in H, this means that $\mathbb{E}(X_n - X)^2 \to 0$. By Jensen's inequality this implies that also $\mathbb{E}(X_n - X) \to 0$ and since $\mathbb{E}(X_n) = 0$ for every n this gives that $\mathbb{E}(X) = 0$.

4 Observe that $T(\alpha X + \beta Y) = \alpha X + \beta Y - \mathbb{E}(\alpha X + \beta Y) = \alpha(X - \mathbb{E}(X)) + \beta(Y - \mathbb{E}(Y)) = \alpha T(X) + \beta T(Y)$. So T is a linear operator. Moreover

$$\mathbb{E}(T(X))^2 = \mathbb{E}(X^2 - 2X\mathbb{E}(X) + \mathbb{E}(X)^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \leq \mathbb{E}(X^2).$$

This gives that $\|T(X)\| \leq \|X\|$ for every $X \in H$ (recall that $\|X\| = [\mathbb{E}(X)^2]^{1/2}$). Therefore the operator is bounded. The norm of the operator $\|T\|$ is less or equal than 1. Now we observe that if $\mathbb{E}(X) = 0$, then $T(X) = X$, so T is the identity on the space C. This implies that $\|T\|$ cannot be less than 1, and then $\|T\| = 1$.

5 By definition of the scalar product in H we have that

$$\mathbb{E}((X - \mathbb{E}(X))k) = k\mathbb{E}(X) - k\mathbb{E}(X) = 0.$$

In particular this implies that $C^\perp \supset \{k \in \mathbb{R}\}$ (the orthogonal space of C contains the set of all constants). Assume now that $Y \in C^\perp$ is not constant. Then $Y - \mathbb{E}(Y) \in C$. On the other hand since $C^\perp$ is a vectorial space also $Y - k \in C^\perp$ for every constant k, then also for $k = \mathbb{E}(Y)$. This implies that $Y - \mathbb{E}(Y) \in C \cap C^\perp$, and then $Y - \mathbb{E}(Y) = 0$, which means that Y is constant.

6 Since $C^\perp$ is the space of constants, and $X - \mathbb{E}(X) \in C$, this implies that $X = X - \mathbb{E}(X) + \mathbb{E}(X)$, that is $X - \mathbb{E}(X)$ is the projection of X in the space C and $\mathbb{E}(X)$ is the projection of X on $C^\perp$. Therefore $k = \mathbb{E}(X)$.