Functional Analysis

Exam- 20 December, 2021- 12.30-14 (90 minutes)

Exercise 1.

1. Give the definition of absolutely continuous measure and of singular measure (with respect to the Lebesgue measure in $\mathbb{R}$).
2. State the (Lebesgue-)Radon-Nikodym theorem.
3. Consider the functions

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} - xe^{-x} & x \geq 0 \end{cases} \qquad G(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{3} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

and let μ_F, μ_G the Borel measures which have these functions as their cumulative distribution functions (so $\mu_F(a,b] = F(b) - F(a)$ and $\mu_G(a,b] = G(b) - G(a)$).

(a) Are the measures finite? In case compute $\mu_F(\mathbb{R})$ and $\mu_G(\mathbb{R})$.

(b) Is the measure μ_F absolutely continuous or is it singular with respect to Lebesgue measure? Find, if it exists, the density associated to this measure.

(c) Is the measure μ_G absolutely continuous or is it singular with respect to Lebesgue measure? Find, if it exists, the density associated to this measure.

Exercise 2.

1. State the orthogonal projection theorem in Hilbert spaces.
2. Let M^2 the space of random variables with finite second moment and consider the subspace

$$C = \{X \in M^2 \, | X \text{ is equal to a constant almost surely}\}.$$

Compute the orthogonal space $C^\perp$ of C. Compute the orthogonal space $(C^\perp)^\perp$.

3. Let $X \in M^2$. Compute the orthogonal projection of X on $C^\perp$.
4. Let $Y \in M^2$ a normal random variable (with mean 0 and variance 1). Let $X \in M^2$. Find $\lambda, \mu \in \mathbb{R}$ such that

$$\mathbb{E}[(X - \lambda Y - \mu)^2] = \min_{a,b} \mathbb{E}[(X - aY - b)^2].$$

Compute this minimal value.

Sketch of solutions

Solution 1.

1. See notes, Definition 2.38.

2. See notes, Theorem 2.32.

3. (a) Since $\mu_F(\mathbb{R}) = \sup F - \inf F = 1, \mu_G(\mathbb{R}) = \sup G - \inf G = 1$, the two measures are finite (and are probability measures).

   (b) Note that F is continuous, then μ_F is absolutely continuous with respect to Lebesgue. The density of μ_F is a nonnegative function f such that $\mu_F(a,b) = \int_a^b f(x)dx$. Since $\mu_F(a,b) = 0$ for every $a, b < 0$, we get that $f(x) = 0$ for all $x < 0$. Moreover since for $x > 0$

   $$\mu_F(0,x) = 1 - e^{-x} - xe^{-x} = \int_0^x f(t)dt$$

   we get, by the fundamental theorem of integral calculus, that

   $$f(x) = (1 - e^{-x} - xe^{-x})' = xe^{-x}.$$

   Therefore the density of μ_F is $f(x) = xe^{-x}\chi_{(0,+\infty)}(x)$.

   (c) Note that $\mu_G\{0\} = G(0) - \lim_{x\to 0^-} G(x) = \frac{1}{3}$ and $\mu_G\{1\} = G(1) - \lim_{x\to 1^-} G(x) = 1 - \frac{1}{3} = \frac{2}{3}$. Therefore μ_G cannot be absolutely continuous. Moreover, $\mu_G(\mathbb{R} \setminus \{0,1\}) = \mu_G(\mathbb{R}) - \mu_G\{0\} - \mu_G\{1\} = 0$. Therefore μ_G is singular with respect to the Lebesgue measure (so it has no density) and moreover $\mu_G = \frac{1}{3}\delta_0 + \frac{2}{3}\delta_1$.

   μ_F is a Gamma distribution whereas μ_G is a binomial distribution.

Solution 2.

1. See notes, Theorem 4.7.

2. C is the space of constant random variables. If $X \in C^\perp$ then $< X, 1 >= \mathbb{E}(X \cdot 1) = 0$, which means that $\mathbb{E}(X) = 0$. On the other hand if $\mathbb{E}(X) = 0$, then $< X, a >= \mathbb{E}(X \cdot a) = 0$ for every constant random variable a. Therefore $C^\perp = \{X \in M^2, \mathbb{E}(X) = 0\}$.

   Now observe that $C \subseteq (C^\perp)^\perp$ since if $c \in C$, then $< X, c >= \mathbb{E}(X \cdot c) = 0$ for every $X \in C^\perp$. Let now $Y \in (C^\perp)^\perp$ which is not constant. Then by definition $< X, Y >= \mathbb{E}(X \cdot Y) = 0$ for every $X \in C^\perp$. Note that since Y is not constant,

$Y - \mathbb{E}(Y) \neq 0$ and moreover $Y - \mathbb{E}(Y) \in C^\perp$, since $\mathbb{E}(Y - \mathbb{E}(Y)) = 0$. Therefore, $< Y - \mathbb{E}(Y), Y >= \mathbb{E}((Y - \mathbb{E}(Y)) \cdot Y) = 0$. Note that $\mathbb{E}((Y - \mathbb{E}(Y)) \cdot Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \mathbb{E}(Y - \mathbb{E}(Y))^2 = 0$, which means that $Y \equiv \mathbb{E}(Y)$, so that Y is constant. This implies that $(C^\perp)^\perp = C$.

3. By the orthogonal projection theorem, every X can be written uniquely as an element of C (which is the orthogonal projection of X on C) and an element of $C^\perp$ (which is the orthogonal projection of X in $C^\perp$). First of all, we compute the orthogonal projection of X on C. A orthonormal basis of C is given by the random variable which is identically equal to 1. Then the orthogonal projection of X on C is $< X, 1 >= \mathbb{E}(X \cdot 1) = \mathbb{E}(X)$. Since $X = \mathbb{E}(X) + (X - \mathbb{E}(X))$, we get using the orthogonal projection theorem, that $X - \mathbb{E}(X)$ is the orthogonal projection of X on $C^\perp$.

4. We have to compute the orthogonal projection of X on the space generated by $1, Y$, that is on the space $S = \{Z \in M^2 \mid Z = aY + b\}$. Note that $\{1, Y\}$ is a orthonormal basis of S since $\mathbb{E}(Y \cdot 1) = \mathbb{E}(Y) = 0$ and $\mathbb{E}(1^2) = \mathbb{E}(Y^2) = 1$. Therefore the orthogonal projection of X is given by the random variable $Z \in S$ defined as
$$Z = \mathbb{E}(X)1 + \mathbb{E}(XY)Y.$$
In particular $\lambda = \mathbb{E}(XY)$ and $\mu = \mathbb{E}(X)$. Finally
$$\begin{aligned} & \mathbb{E}(X - \mathbb{E}(X)1 - \mathbb{E}(XY)Y)^2 \\ = & \mathbb{E}(X^2) + (\mathbb{E}(X))^2 + (\mathbb{E}(XY))^2\mathbb{E}(Y^2) - 2(\mathbb{E}(X))^2 - 2(\mathbb{E}(XY))^2 + 2\mathbb{E}(XY)\mathbb{E}(X)\mathbb{E}(Y) \\ = & \mathbb{E}(X^2) - (\mathbb{E}(X))^2 - (\mathbb{E}(XY))^2 = Var(X) - Cov(X, Y)^2. \end{aligned}$$