Functional Analysis

Exam- December 20, 2023- 90 minutes

Exercise 1.

1. Give the definition of absolutely continuous measure with respect to Lebesgue. Recall the characterization (in terms of the density function..) of Borelian σ-finite measures on $\mathbb{R}$ which are absolutely continuous with respect to Lebesgue.

2. Let
$$f(x) = \begin{cases} e^x & -\infty < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & \end{cases}$$
Is f the density of an absolutely continuous measure μ? If the answer is positive, compute $\mu(\mathbb{R})$.

3. Give the definition of singular measure with respect to Lebesgue and provide an example.

Exercise 2.

1. State the orthogonal projection theorem in a general Hilbert space H.

2. Using the notion of orthonormal basis, state the formula to compute the orthogonal projection of a generic element $h \in H$ to a closed subspace $V \subseteq H$.

Let $H = M^2(\Omega, \mathbb{P}, \mathcal{F})$ the space of random variables with bounded second moment. For $\mathcal{G}$ a σ-algebra in Ω strictly contained in $\mathcal{F}$, we define $V = M^2(\Omega, \mathbb{P}, \mathcal{G})$ the closed subspace of H which contains $\mathcal{G}$-measurable random variables with bounded second moment.

4 Let $\mathcal{G} = \{\emptyset, \Omega)$ and compute the orthogonal projection of a random variable $X \in H$ in the closed subspace $M^2(\Omega, \mathbb{P}, \mathcal{G})$.

5 Let $A \in \mathcal{F}$, and $\mathcal{G} = \{\emptyset, \Omega, A, \Omega \setminus A)$. Compute the orthogonal projection of a random variable $X \in H$ in the closed subspace $M^2(\Omega, \mathbb{P}, \mathcal{G})$.
Hint: the space $M^2(\Omega, \mathbb{P}, \mathcal{G}) = \{a\chi_A + b$, with $a, b \in \mathbb{R}\}$, where $\chi_A(\omega)$ is the random variable which assumes values 1 for $\omega \in A$ and 0 elsewhere.

Sketch of solutions

Solution 1.

2 Observe that $f \geq 0$, moreover $\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^{0} e^x dx + \int_0^1 (1-x)dx = [e^x]_{-\infty}^0 + [x - \frac{x^2}{2}]_0^1 = 1 + 1 - \frac{1}{2} = \frac{3}{2}$. So $f \in L^1(\mathbb{R})$. Finally $\mu(\mathbb{R}) = \int_{\mathbb{R}} f(x)dx = \frac{3}{2}$.

Solution 2.

4 Since $M^2(\Omega, \mathbb{P}, \mathcal{G})$ is the space of constant random variables, and $X - \mathbb{E}(X)$ is orthogonal to every constant, we conclude that the projection of X in $M^2(\Omega, \mathbb{P}, \mathcal{G})$ is the constant $\mathbb{E}(X)$.

5 $M^2(\Omega, \mathbb{P}, \mathcal{G})$ is a 2 dimensional space generated by $1, \chi_A$. We orthonormalize this basis, and obtain $X_1 = 1$, $X_2 = \frac{\chi_A - \mathbb{E}(\chi_A)}{\sqrt{\mathbb{E}[(\chi_A - \mathbb{E}(\chi_A))^2]}}$. Since $\mathbb{E}(\chi_A) = \mathbb{P}(A)$, we obtain

$$X_2 = \frac{\chi_A - \mathbb{P}(A)}{\sqrt{\mathbb{P}(A)(1-\mathbb{P}(A)}}.$$

So the projection of a random variable X is given by

$$\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X1) + \mathbb{E}(XX_2)X_2 = \mathbb{E}(X) + \frac{\mathbb{E}(X\chi_A) - \mathbb{E}(X)\mathbb{P}(A)}{\mathbb{P}(A)(1-\mathbb{P}(A))}(\chi_A - \mathbb{P}(A)).$$