OBSERVATION (Exercise)

(1) Let $A_i \subseteq \mathbb{R}^n$ $\quad \dim_{\mathcal{H}} A_i = c_i \quad \forall i$

then $A = \cup_i A_i$ satisfies $\dim_{\mathcal{H}}(A) = \sup_i c_i$

proof.

Let $c > c_i \ \forall i \Rightarrow \mathcal{H}^c(A_i) = 0 \Rightarrow \mathcal{H}^c(A) \leq \sum_i \mathcal{H}^c(A_i) = 0$

so $\forall c > \sup_i c_i \quad \mathcal{H}^c(A) = 0$

Let $c < \sup_i c_i \Rightarrow \exists k$ such that $c < c_k \Rightarrow \mathcal{H}^c(A_k) = +\infty$

$\Rightarrow \mathcal{H}^c(A) \geq \mathcal{H}^c(A_k) = +\infty$

$\Rightarrow \dim_{\mathcal{H}} A = \sup_i c_i$

(2) if $c_i < n \ \forall i$ and $\sup_i c_i = n$

$\dim_{\mathcal{H}}(A) = n \qquad \mathcal{H}^n(A) = 0$

3 Construction "à la Cantor"

Fix $0 < \alpha < 1$.

In $\mathbb{R}$ consider family K_n of compact sets contained in $[0,1]$ such that $K_0 = [0,1]$

$K_n \subseteq K_{n-1}$, K_n is the union of 2^n disjoint intervals of length δ_n (Cantor $\delta_n = 1/3^n$)

where δ_n is a sequence such that

1) $\delta_n < \frac{1}{2}\delta_{n-1}$

2) $\lim_n 2^n \delta_n^\alpha = 0$

3) $\lim_n 2^n \delta_n^\beta = +\infty \quad \forall\, 0 < \beta < \alpha$

for Cantor this NOT TRUE for $\alpha = \log_3 2$!

ex: $\delta_n = 2^{-\frac{n}{\alpha} + \sqrt{n}}$ check that it works! (1, 2, 3 satisfied!)

$K = \bigcap K_n$ is a compact set such that $\underbrace{\mathcal{H}^\alpha(K) = 0}$ $\mathcal{H}^\beta(K) = +\infty$ $\forall \beta < \alpha$.

it comes from the fact that $2^n \delta_n^\alpha \to 0$

this comes more or less $\lim_n 2^n \delta_n^\beta \to +\infty$ (but to be proved!!)

[based on FROSTMAN'S LEMMA]

If this is true

$\dim_{\mathcal{H}}(K) = \alpha$

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4. $\forall \alpha \in (0,1)$ $\exists K_\alpha$ with

$$\dim_{\mathcal{H}}(K_\alpha) = \alpha$$

$\forall \alpha = 1 - \frac{1}{n}$ K_n $\rightarrow$ $\bigcup_n K_n$ is a set of Hausdorff dim 1, but $\mathcal{H}^1(\bigcup_n K_n) = 0$. (by point 2).

⑤ C_1, C_2 2 sets in $\mathbb{R}$ with

$$\dim_{\mathcal{H}} C_1 = \dim_{\mathcal{H}} C_2 = 1$$

$$\mathcal{H}^1(C_1) = 0 = \mathcal{H}^1(C_2)$$

$C_1 \times C_2 \subseteq \mathbb{R}^2$

$$\dim_{\mathcal{H}}(C_1 \times C_2) \geq \underbrace{\dim_{\mathcal{H}} C_1 + \dim_{\mathcal{H}} C_2}_{= 2}$$

general fact
(depends on the fact that
$\operatorname{diam}(A, \operatorname{diam} B \leq \delta \Rightarrow \operatorname{diam} A \times B \leq \sqrt{2}\delta$

$C_1 \times C_2$ is a purely 1-unrectiable set.

If M is a $\mathcal{C}^1$ 1-dim manifold $\rightarrow$ it's a $\mathcal{C}^1$ curve in $\mathbb{R}^2$

take $M = \{(t, \gamma(t)) \quad \gamma \in \mathcal{C}^1\}$ (at least locally)

$$M \cap (C_1 \times C_2) \subseteq \{(t, \gamma(t)) \quad t \in C_1\}$$

$$\mathcal{H}^1(M \cap (C_1 \times C_2)) \leq \mathcal{H}^1\{(t, \gamma(t)),\ t \in C_1\} \leq$$

$$\leq \mathcal{H}^1(C_1) \cdot \sqrt{1 + |\gamma'|_\infty^2} = 0$$