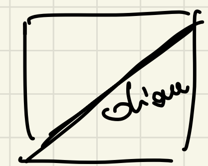
Second part of the proof $\mathcal{H}^n = \mathcal{L}$ in $\mathbb{R}^n$

2) we prove that $\mathcal{H}^n << \mathcal{L}$

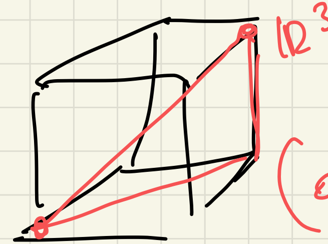
Q cube in $\mathbb{R}^n = (a,b)^n$ $\quad b > a$

$\text{diam } Q = \sqrt{n}\ \text{edge}(Q)$



$\mathbb{R}^2$

$\text{diag} = \sqrt{2}\ \text{edge}$



$(\text{diam } Q)^2 = (\text{edge})^2 + 2(\text{edge})^2$
$= 3(\text{edge})^2$

$$|Q| = (\text{edge } Q)^n = \left(\frac{\text{diam } Q}{n^{n/2}}\right)^n$$

$E \in \mathcal{B}(\mathbb{R}^n)$ $\quad Q_i$ $\text{diam } Q_i \le \delta$ $\quad E \subseteq \bigcup_i Q_i$

$$\mathcal{H}^n_\delta(E) \le \frac{\omega_n}{2^n} \sum_i (\operatorname{diam} Q_i)^n$$

$E \subseteq \bigcup_i Q_i$

$\operatorname{diam} Q_i \le \delta$

$(\operatorname{diam} Q_i)^n = \frac{|Q_i|}{n^{n/2}}$

$$= \frac{\omega_n}{2^n} \sum_i \frac{|Q_i|}{n^{n/2}} =$$

$$= \frac{\omega_n}{2^n n^{n/2}} \sum_i |Q_i|$$

$E \subseteq \bigcup_i Q_i$

$\operatorname{diam} Q_i \le \delta$

taking infimum among all possible covering of E by cubes of diam $\le \delta$

$$\mathcal{H}^n_\delta(E) \le \frac{\omega_n}{2^n n^{n/2}} |E|$$

$$\Rightarrow \mathcal{H}^n(E) \le \frac{\omega_n}{2^n} n^{n/2} |E|$$

$$\mathcal{H}^n \ll \mathcal{L}$$

3 Now I want to prove $\mathcal{H}^n(E) \le |E|$

Fix $E \in \mathcal{B}(\mathbb{R}^n)$

fix $\varepsilon > 0$ fix $\delta > 0$

$\Rightarrow \underline{\exists\, Q_i}$ $E \subseteq \bigcup_i Q_i$ diam $Q_i \le \delta$

$$|E| + \varepsilon \ge \sum_i |Q_i| \ge |E|$$

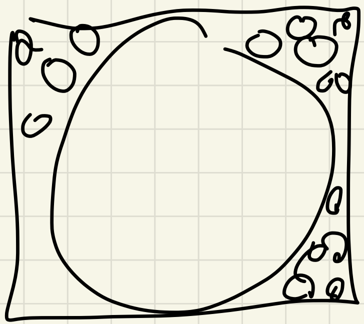
(choose all Q_i to be open)

$\forall i \; \exists\, B_i^k$ family of disjoint balls

by Vitali Besikovitch

diam $B_i^k \le \delta$

$$\left|Q_i \setminus \bigcup_k B_i^k\right| = 0 \qquad B_i^k \cap B_j^k = \emptyset$$



$\forall x \in Q_i$ take all $r > 0$ such that $B(x,r) \subseteq Q_i$, $2r \leq \delta$

$$\mathcal{F} = \{ B(x,r) \quad x \in Q_i, \ 2r \leq \delta, \ B(x,r) \subseteq Q_i \}$$

fine cover by balls

apply Vitali–Besicovitch ..

$\forall i \quad \exists B_i^k \subset Q_i \qquad |Q_i \setminus \bigcup_k B_i^k| = 0$

$\operatorname{diam} B_i^k \leq \delta \qquad B_i^k \cap B_j^{\ell} = \emptyset \quad h \neq \ell$

$\mathcal{H}^n(Q_i \setminus \bigcup_k B_i^k) = 0$ (since $\mathcal{H}^n << \mathcal{L}$)

$\mathcal{H}^n_\delta(Q_i \setminus \bigcup_k B_i^k) = 0 \qquad E \subseteq \bigcup Q_i$

$$\mathcal{H}^n_\delta(E) \leq \Sigma_i \mathcal{H}^n_\delta(Q_i) = \Sigma_i \mathcal{H}^n_\delta(\bigcup_k B_i^k) \leq$$

$$\leq \Sigma_i \Sigma_k \mathcal{H}^n_\delta(B_i^k) \leq \Sigma_i \Sigma_k \frac{\omega_n}{2^n} (\operatorname{diam} B_i^k)^n$$

since B_i^k has diam $\leq \delta$.

$$\mathcal{H}^n_\delta(E) \le \dots \quad \le \sum_i \sum_k \underbrace{\frac{\omega_n}{2^n}(\operatorname{diam} B_i^k)^n}_{|B_i^k|} =$$

$$= \sum_i \sum_k \underbrace{|B_i^k| = \sum_i}_{|Q_i \setminus \cup_i B_i^k| = 0} |Q_i| \le |E| + \varepsilon$$

$$\mathcal{H}^n(E) \le |E| + \varepsilon \qquad \forall \varepsilon$$

$$\downarrow$$

$$\mathcal{H}^n(E) \le |E| \qquad \square \qquad (\mathcal{H}^n = \mathcal{L})$$

$\mathcal{H}^k$ in $\mathbb{R}^n$ for $k < n$ $k \in \mathbb{N}$

$M \subseteq \mathbb{R}^n$ is a C^1 k-dimensional (sub)manifold

(1) $\forall x \in M$ $\exists U$ open $x \in U$
(neighborhood of x)

$\exists f : V \subseteq \mathbb{R}^k \to \mathbb{R}^n$ such that
open

- $f \in C^1$
- f is injective
- $D_x f = \left(\partial_{x_j} f^i \right)_{\substack{i=1\dots n \\ j=1\dots k}}$ is injective

$f(V) = M \cap U$

f is a parametrization of M nearby x

(2) M can be covered by a countable number of param.

$\boxed{\text{Proposition}}$ $k < n$

$T : \mathbb{R}^k \to \mathbb{R}^n$ linear map.

T^* adjoint : $\mathbb{R}^n \to \mathbb{R}^n$

$\forall\, x \in \mathbb{R}^n$
$y \in \mathbb{R}^k$

$y \cdot (T^* x) = (Ty) \cdot x$

scalar product in $\mathbb{R}^k$

scalar product in $\mathbb{R}^n$

$J(T) =$ jacobian of $T = \sqrt{\det(T^* T)}$

$\forall\, A \subseteq \mathbb{R}^k$ Borel $\mathcal{H}^k(T(A)) = J(T)\, \mathcal{H}^k(A)$

$= J(T)\, |A|$

idea is to consider a rotation in $\mathbb{R}^n$

such that up to this rotation

$$\operatorname{Im}(T) \subseteq \{(x_1 \ldots x_n) \in \mathbb{R}^n \quad x_i = 0 \ i > k\} =$$

$$= \mathbb{R}^k \times \{0\}^{n-k}$$

then apply the change of variable

formula in $\mathbb{R}^k$.

<u>Proposition</u> $V \subseteq \mathbb{R}^k$ open (AREA FORMULA)

$f : V \to \mathbb{R}^n$ injective

C^1, $(D_x f)$ is injective

(AREA FORMULA)

$\forall A \subseteq V$ Borel

$$\mathcal{H}^k f(A) = \int_A \underbrace{J(D_x f)} \, dx$$

$\varphi \geq 0$

$$\int_V \phi(f(x)) \, J(D_x f) \, dx = \int_{f(V)} \phi(y) \, d\mathcal{H}^k(y)$$

$\mathcal{H}^k$ on C^1 k-dimensional manifolds of $\mathbb{R}^n$ coincides with the classical "surface measure".

<u>Def</u> $E \subseteq \mathbb{R}^n$ (Borel)

E is countable k-rectifiable if $\exists$ countable family of C^1 k-dim submanifolds M_i such that $\mathcal{H}^k(E \setminus \bigcup_i M_i) = 0$

E is k-rectifiable if it is countable k-rectif. and $\mathcal{H}^k(E) < +\infty$.

E is purely k-unrectifiable if

$\mathcal{H}^k(E \cap M) = 0 \quad \forall M \ C^1 \ k$-dimens. submanifold of $\mathbb{R}^n$.