funzioni "elementari"   funzioni polinomiali, polinomiali fratte, radicali, esponenziali, logaritmiche, trigonometriche

$x \longmapsto x^n$

$x \longmapsto \sqrt[n]{x}$

$x \longmapsto a^x$ con $a \in (0, +\infty)$ $a \neq 1$

$a^y = x$

$x \longmapsto \dfrac{1}{x^n}$

$x \longrightarrow \log x$ $a \in (0, +\infty)$ $a \neq 1$ $x \longmapsto \underbrace{\log_a x}_{=y}$

$x \longrightarrow \sin x$

$x \longrightarrow \cos x$

$x \longrightarrow \operatorname{tg} x$

operazione di composizione tra funzioni ci permette, a partire dalle funzioni elementari, di costruire tutte le altre funzioni.

$$x \longmapsto \sin(x^2) \qquad f(x) = \sin(x^2)$$

$$x \overset{g}{\longmapsto} x^2 \overset{h}{\longrightarrow} \sin(x^2)$$

$g: x \longrightarrow x^2$

$g(x) = x^2$

$h: x \longrightarrow \sin x$

prima applico f polinomiale g e poi la f. trigonometrica h $f(x) = h \circ g(x)$

g è la prima operaz e h la seconda

$$f(x) = \boxed{h \circ g}(x) = h(g(x))$$

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend left, "f"] & x^2 \arrow[r, "h"] & \sin(x^2) \end{tikzcd}$$

$g: x \rightarrow x^2$ $\qquad$ $h: x \rightarrow \sin x$

$f(x): x \rightarrow \sin(x^2)$

PER LA COMPOSIZIONE se cambio ordine con cui eseguo le operazioni, il risultato CAMBIA

SE CAMP

se cambio ordine

(prima h e poi g)

$g: x \longrightarrow x^2$

$h: x \longrightarrow \sin x$

$$x \xrightarrow{h} \sin x \xrightarrow{g} (\sin x)^2$$

$$g \circ h(x) = g(h(x)) = (\sin x)^2$$

$$\sin(x^2) \neq (\sin x)^2$$

$x = \sqrt{\frac{3}{2}\pi}$

$\sin\left[\left(\sqrt{\frac{3}{2}\pi}\right)^2\right] = \sin\left(\frac{3}{2}\pi\right) = -1$

$\left[\sin\left(\sqrt{\frac{3}{2}\pi}\right)\right]^2 > 0$

$g: x \longrightarrow x^4$ $\qquad$ $h: x \longrightarrow \log x$

$$f(x) = h \circ g(x) = h(g(x)) = \log(x^4)$$

$$f: x \xrightarrow{g} x^4 \xrightarrow{h} \log(x^4)$$

D: ?

$x^4 > 0$

$\Downarrow$

$x \neq 0$

DOMINIO $= (-\infty, 0) \cup (0, +\infty)$

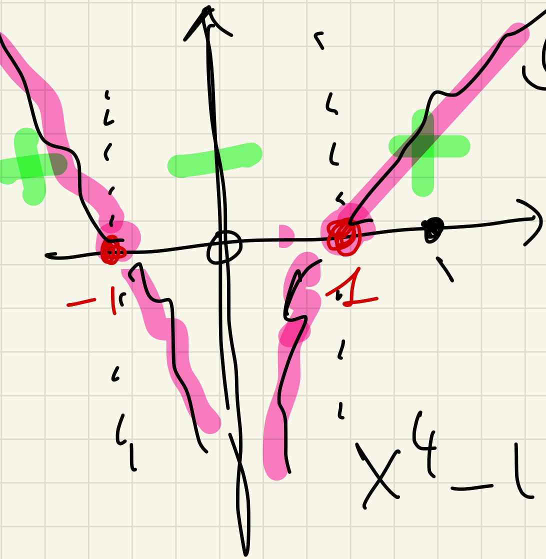
$= \{x \in \mathbb{R} \quad x \neq 0\}$

$$f(-x) = \log(-x)^4 = \log x^4 = f(x)$$

f PARI

segno $f(x) \ge 0$?

$a = \lg e^a$



$$\lg(x^4) \ge 0 = \lg 1$$

$$\boxed{\lg(x^4) = 4\lg|x|}$$

$$\downarrow$$

$$x^4 \ge 1$$

$$x^4 - 1 \ge 0$$

$$(x^2-1)\underbrace{(x^2+1)}_{>0} \ge 0$$

$x^2+1>0 \quad \forall x \in \mathbb{R}$

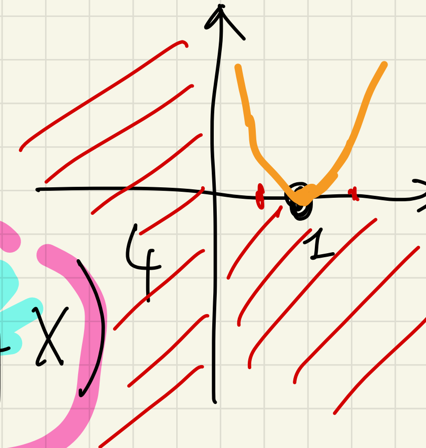
$f(1)=0$

$f(-1)=0$

$x^2-1 \ge 0 \Rightarrow$?

$x \ge 1$ o $x \le -1$

$h : x \to \lg x \qquad g : x \to x^4$



$$x \xrightarrow{h} \lg x \xrightarrow{g} (\lg x)^4$$

$$g \circ h(x) = g(h(x)) = (\lg x)^4$$

$D = \{x > 0\} = (0, +\infty)$

ricorda. non è simmetrica

segno $f(x) \geq 0 \qquad (\lg x)^4 \geq 0 \quad \forall x \in D$

$f(x) > 0 \quad \forall x \in D \quad x \neq 1 \qquad f(1) = 0$

FUNZIONE INVERSA

$g = f^{-1}$ $\cancel{\left(\neq \frac{1}{f}\right)}$

$f : D_f \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ D_f dominio di f

Definizione

diciamo che f ammette funzione inversa

(f è INVERTIBILE)

se esiste $g : D_g \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ D_g dominio di g

FUNZIONE INVERSA di f

tale che

$$f \circ g(x) = f(g(x)) = x \quad \forall x \in D_g$$

$$g \circ f(x) = g(f(x)) = x \quad \forall x \in D_f$$

es: $f : x \longrightarrow x^3$

$\mathbb{R} \longrightarrow \mathbb{R}$

$D_f = \mathbb{R}$

$f^{-1}(x) = \sqrt[3]{x}$ $\cancel{\left(\neq \frac{1}{f(x)} = \frac{1}{x^3} \right)}$

$g(x) = x^{1/3}$

$\exists$ g inversa

$g(f(x)) = x \rightarrow g(x^3) = x$

$f(g(x)) = x \rightarrow (g(x))^3 = x$

$g(f(x))$

$x \longrightarrow f(x) \xrightarrow{g} g(f(x))$

$x \longrightarrow x^3 \longrightarrow g(x^3)$

$f(g(x))$

$x \longrightarrow g(x) \xrightarrow{f} (g(x))^3$

$f: x \to x^3$

$D_f = \mathbb{R}$

$\boxed{g: x \to x^{\frac{1}{3}} = \sqrt[3]{x}}$

$D_g = \mathbb{R}$

$f \circ g(x) = f(\underbrace{g(x)}) = x$

$$x \xrightarrow{g} x^{\frac{1}{3}} \longrightarrow (x^{1/3})^3 = x$$

$g \circ f(x) = g(f(x)) = x$

$$x \xrightarrow{f} x^3 \xrightarrow{g} (x^3)^{\frac{1}{3}} = x$$

$f: x \longrightarrow x^2$

$D_f = \mathbb{R}$

$4 \longrightarrow 2$

$\boxed{g: x \longrightarrow \sqrt{x}}$

$D_g = [0, +\infty)$

(g è una funzione
$\forall x \in [0, +\infty]$ $\exists!$ $g(x)$
RADICE POSITIVA

$g \circ f(x) = |x| \quad \forall x \in \mathbb{R}$

$x \xrightarrow{f} x^2 \xrightarrow{g} \sqrt{x^2} = |x|$

$(-2) \longrightarrow (-2)^2 = 4 \longrightarrow \sqrt{4} = 2$

$f : x \longrightarrow x^2$

$D_f = \mathbb{R}$

$g : x \longrightarrow \sqrt{x}$ $\quad D_g = [0, +\infty]$

$$f \circ g(x) = f(g(x)) = (\sqrt{x})^2 = x \quad \forall x \geq 0$$

$x \in D_g \downarrow$ $\boxed{x \geq 0}$

$$\underset{0}{\overset{x}{\vee|}} \longrightarrow \sqrt{x} \longrightarrow (\sqrt{x})^2 = \underset{0}{\overset{x}{\vee|}}$$

$$\begin{cases} f \circ g(x) = x \quad \forall x \in D_g \quad \text{cioè } \forall x \geq 0 \\ g \circ f(x) = |x| \quad \forall x \in D_f \quad \text{cioè } \forall x \in \mathbb{R} \end{cases}$$

(~~f e g sono inverse~~

se NON considero il dominio naturale di f ma un suo sottoinsieme (cioè $[0,+\infty)$) allora

g è l'inverso di f

e f è l'inverso di g

$$f : [0,+\infty) \to \mathbb{R}, \quad x \longmapsto x^2 \qquad g : [0,+\infty) \to \mathbb{R}, \quad x \longmapsto \sqrt{x}$$

$$\boxed{a = \lg e^{a} \qquad \forall\, a \in \mathbb{R}}$$

$$\boxed{e^{\lg a} = a \qquad \forall\, a > 0}$$

$f: x \longrightarrow \lg x$ $\qquad$ $g: x \longrightarrow e^{x}$

$D_f = (0, +\infty)$ $\qquad$ $D_g = \mathbb{R}$

$f \circ g(x) = f(g(x)) = \lg(e^{x}) = x \quad \forall\, x \in \mathbb{R}$ $\quad\Big|\quad$ $x \xrightarrow{g} e^{x} \xrightarrow{f} \lg(e^{x})$

$g \circ f(x) = g(f(x)) = e^{\lg x} = x \quad \forall\, x > 0$ $\qquad$ $x \xrightarrow{f} \lg x \xrightarrow{g} e^{\lg x}$

Quali condizioni assicurano che $f: D \to \mathbb{R}$ sia invertibile?

(1) Se f è STRETTAM. MONOTONA (crescente o decrescente) nel suo DOMINIO allora è invertibile

$a > b \quad a, b \in D_f$

$f(a) \underset{<}{>} f(b)$ (strett. mon. crescente)

questa condizione è SUFFICIENTE ma non necessaria!

es $f(x) = \frac{1}{x}$

$D_f = \{x \neq 0\} = (-\infty, 0) \cup (0, +\infty)$

$x \neq 0$

$g(x) = \frac{1}{x}$

f è inversa di se stessa

$f \circ g(x) = x$ $\forall x \neq 0$

$x \xrightarrow{g} \frac{1}{x} \xrightarrow{f} \frac{1}{\frac{1}{x}} = x$

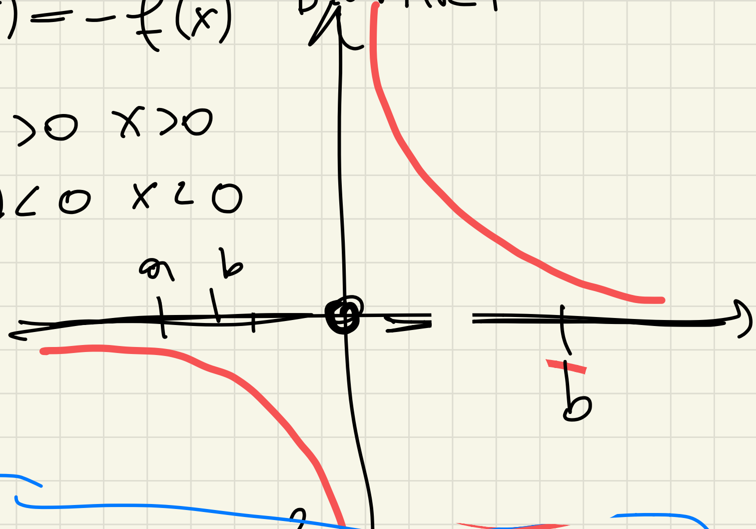
$g \circ f(x) = x$

(la funzione inversa di f è proprio f!)

$f(-x) = -f(x)$ DISPARI

$f(x) > 0 \quad x > 0$

$f(x) < 0 \quad x < 0$



se $a < b \Longrightarrow f(a) > f(b)$?

$f(a) > f(b)$

$f(x) = \frac{1}{x} \qquad x \neq 0$

grafico sono i pti (x, y) con $y = \frac{1}{x}$

$xy = 1$

$a < b \iff f(a) > f(b)$

$a < b < 0$

$0 < a < b$

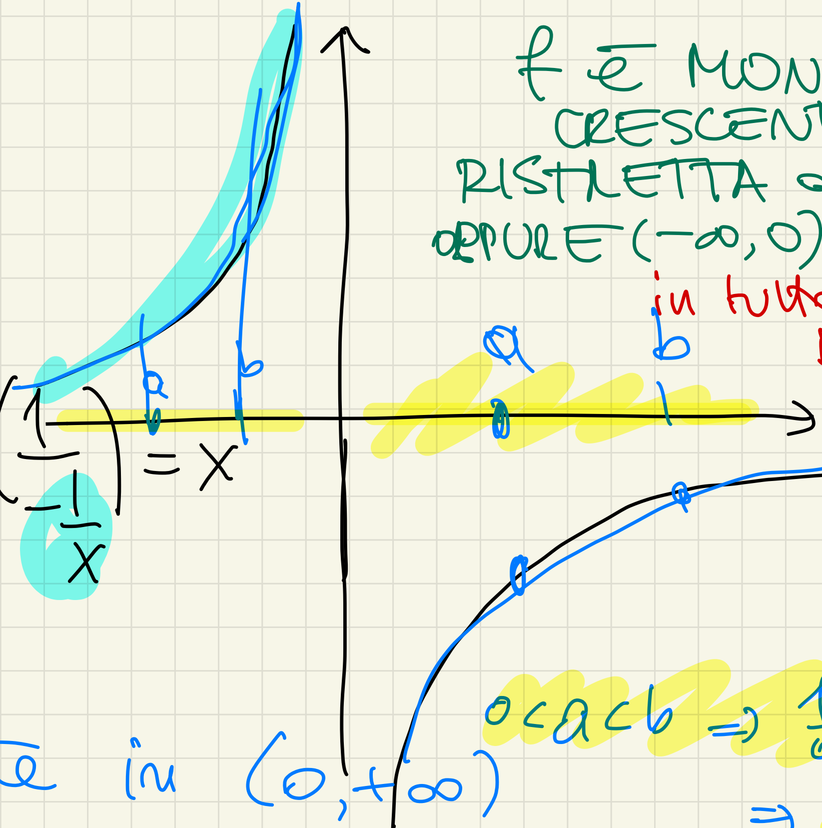
$a, b > 0 \qquad a < b \longrightarrow \frac{1}{a} > \frac{1}{b} \qquad$ OK

$a, b < 0 \qquad \frac{a < b}{\cancel{a}} \qquad 1 \frac{\cancel{a}}{\cancel{a}} > \frac{b}{a} \qquad \frac{1}{b} < \frac{\cancel{b}}{a} \cdot \frac{1}{\cancel{b}} = \frac{1}{a}$

$$f(x) = -\frac{1}{x}$$

$$f^{-1}(x) = -\frac{1}{x}$$

$$x \xrightarrow{f} -\frac{1}{x} \longrightarrow -\left(\frac{1}{-\frac{1}{x}}\right) = x$$



f è MONOTONA CRESCENTE se RISTRETTA a $(0,+\infty)$ OPPURE $(-\infty,0)$ MA non in tutto il DOMINIO

$0<a<b \Rightarrow \frac{1}{a} > \frac{1}{b}$

$\Rightarrow -\frac{1}{a} < -\frac{1}{b}$

f è crescente in $(0,+\infty)$

è crescente in $(-\infty, 0)$

$a<b<0$

$\frac{1}{a} > \frac{1}{b} \Rightarrow -\frac{1}{a} < -\frac{1}{b}$

$a < 0 < b$ $\quad f(a)>0 \quad f(b)<0$

$\cancel{f(a) < f(b)}$

La condizione necessaria e sufficiente per essere invertibile è essere INIETTIVA

cioè $x_1 \neq x_2$ $\quad x_1, x_2 \in D$

$$\Downarrow$$

$$f(x_1) \neq f(x_2)$$

$f(x) = x^2$ NON È INIETTIVA $\quad f(-2) = f(2)$

$-2 \neq 2$

f. TRIGONOMETRICHE non sono invertibili

(perché non sono iniettive

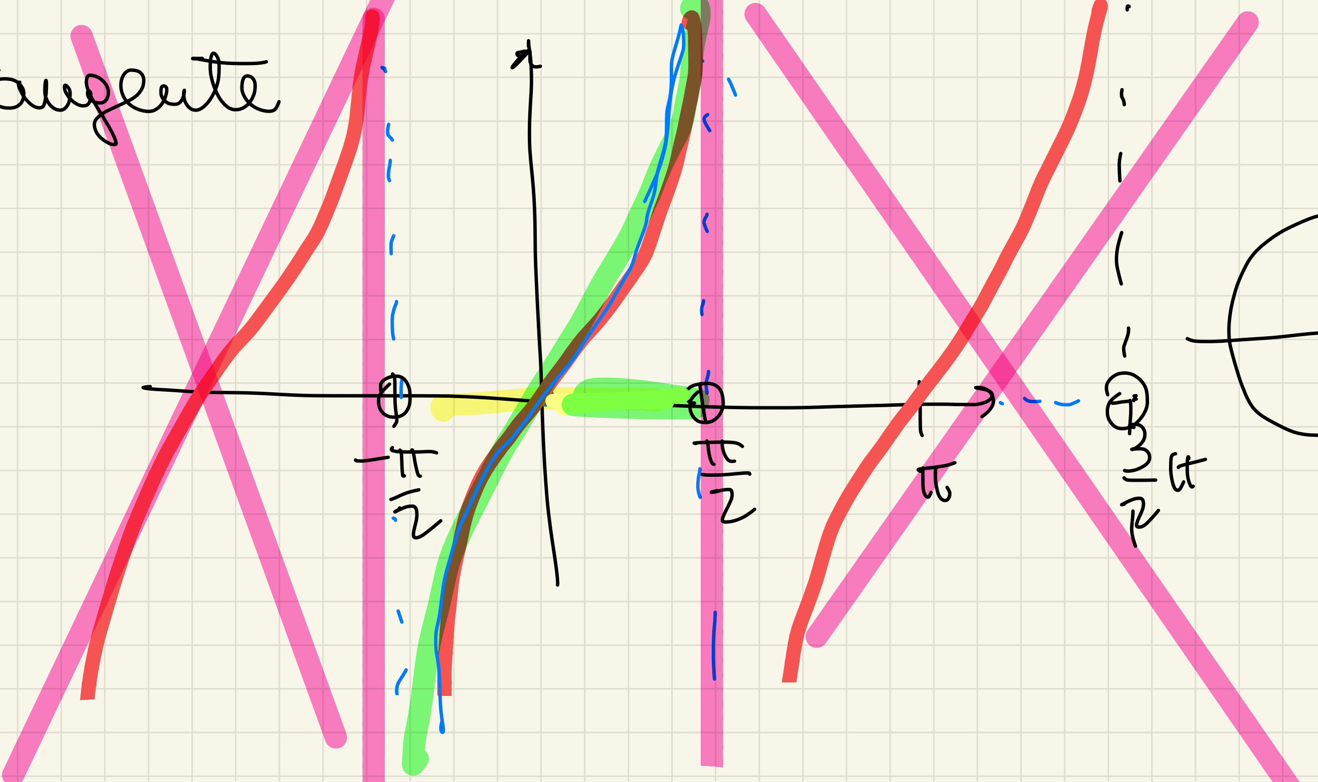
$$\sin(0) = \sin(\pi) = 0 \qquad x_1 = 0 \neq x_2 = \pi$$

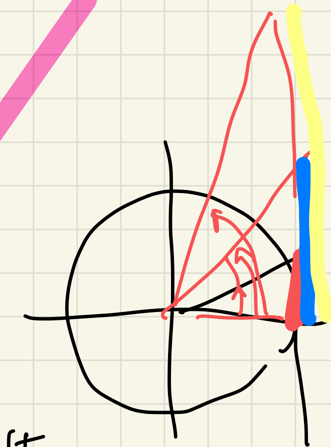
$$x_1 \neq x_2$$

$$f(x_1) = f(x_2)$$

$$\tan\left(\frac{\pi}{4}\right) = 1 = \tan\left(\frac{5}{4}\pi\right)$$

tangente





se io guardo tg x solo per $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(NON È IL DOMINIO NATURALE) è MONOT. strett. CRESC.

$\Rightarrow$ INVERTIBILE

$\operatorname{tg} x$ per $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\qquad \operatorname{tg} x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

$\downarrow$

chiamo $\operatorname{arctg} x$ $=$ ARCO TANGENTE di x

la funzione inversa

$$\operatorname{arctg} x : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\boxed{-\frac{\pi}{2} < \operatorname{arctg} x < \frac{\pi}{2}} \qquad \forall x \in \mathbb{R}$$

$\forall x \in \mathbb{R}$ $\qquad \operatorname{tg}(\operatorname{arctg} x) = x$

$\forall x \in \left(-\frac{\pi}{2}, \pi/2\right)$ $\qquad \operatorname{arctg}(\operatorname{tg} x) = x$

$$\operatorname{tg}(\operatorname{arctg} x) = x \quad \forall x \in \mathbb{R}$$

$$\boxed{\operatorname{arctg}(\operatorname{tg} \underline{\underline{x}}) = x \quad \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$$

~~arctg 0~~

oss. $\operatorname{tg}\frac{\pi}{4} = 1$ $\quad \operatorname{arctg} 1 = \frac{\pi}{4}$ $\quad \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\operatorname{arctg}\left(\operatorname{tg}\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$\operatorname{tg}\left(\frac{5}{4}\pi\right) = 1$$

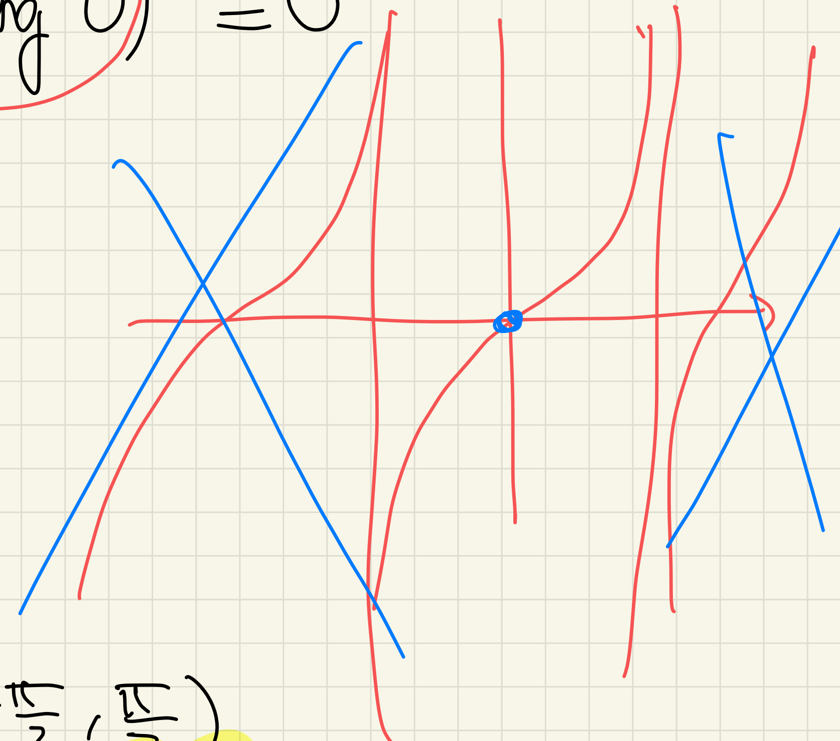
$$\operatorname{arctg}\left(\operatorname{tg}\left(\underline{\frac{5}{4}\pi}\right)\right) = \operatorname{arctg}(1) = \frac{\pi}{4}$$

$x = 0 \qquad \operatorname{tg}(\operatorname{arctg} 0) = 0$

$\operatorname{arctg} 0 = 0$

$\operatorname{arctg} 1 = \frac{\pi}{4}$

$\operatorname{arctg}(-1) = -\frac{\pi}{4}$



$\operatorname{arctg}(x) : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$D = \mathbb{R}$ limitata $\quad -\frac{\pi}{2} < \operatorname{arctg} x < \frac{\pi}{2}$

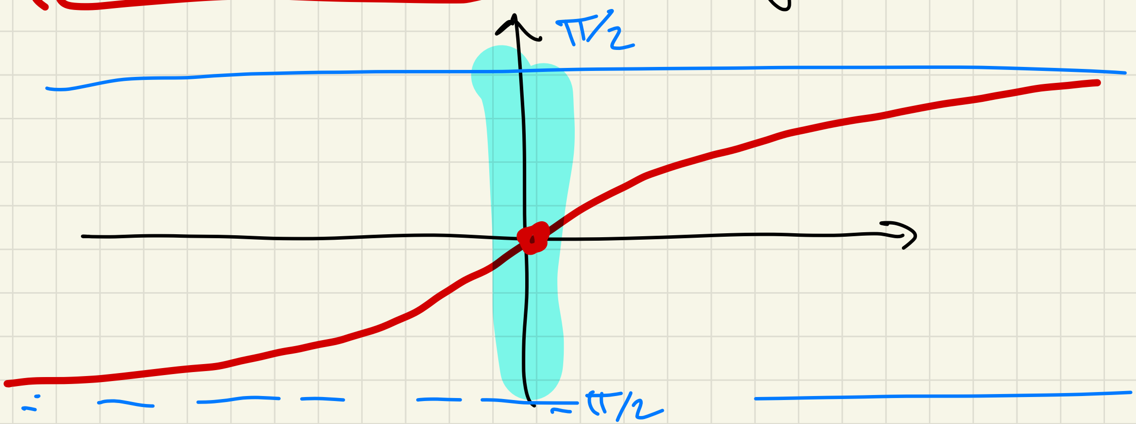
arctg x

- Dominio $\mathbb{R}$
- limitata $-\frac{\pi}{2} < \operatorname{arctg} x < \frac{\pi}{2}$

Non assume mai il valore $\pm\frac{\pi}{2}$

- DISPARI $\operatorname{arctg}(-x) = -\operatorname{arctg} x$

$\boxed{\operatorname{arctg} 0 = 0}$ $\operatorname{arctg}(x)$ è MONOT. CRESC. strett.



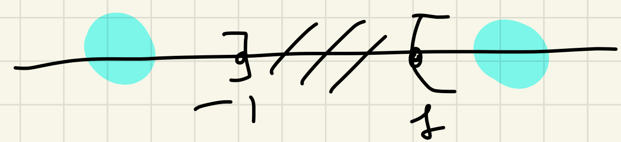
Es. $f(x) = \operatorname{arctg}\left(\sqrt{x^2-1} + x\right)$

D? segno? ev. simmetrie?

$\downarrow$

DOMINIO $\quad x^2 - 1 \geq 0 \Rightarrow x \leq -1, \; x \geq 1$

$(-\infty, -1] \cup [1, +\infty)$



studio simm.

$f(-x) = \operatorname{arctg}\left(\sqrt{(-x)^2-1} + (-x)\right) =$

$= \operatorname{arctg}\left(\sqrt{x^2-1} - x\right) \neq f(x) \,/\, \neq -f(x)$

segno

$f(x) \geq 0$

$$\arctan\left(\sqrt{x^2-1}+x\right) \geq 0 = \arctan 0$$

(arcotangente è POSITIVA sui POSITIVI e NEGATIVA sui NEGATIVI e 0 in 0).

arctg

arctan

(calc. ~~$\tan^{-1}$~~)

$\Updownarrow$

$$\sqrt{x^2-1}+x \geq 0$$

$$\sqrt{x^2-1}+x\geq 0 \qquad x\in(-\infty,-1]\cup[1,+\infty)$$

se $x\in[1,+\infty)$ ho che $\sqrt{x^2-1}\geq 0 \quad x\geq 0$

$$\sqrt{x^2-1}+x>0$$

$\boxed{\text{se } x\in(-\infty,-1]}$

$$\sqrt{x^2-1}+x\geq 0 \Rightarrow \left(\sqrt{x^2-1}\right)^2\geq(-x)^2$$

≥ 0

$$\cancel{x^2}-1\geq\cancel{x^2}$$

$-1\geq 0$ ASSURDO

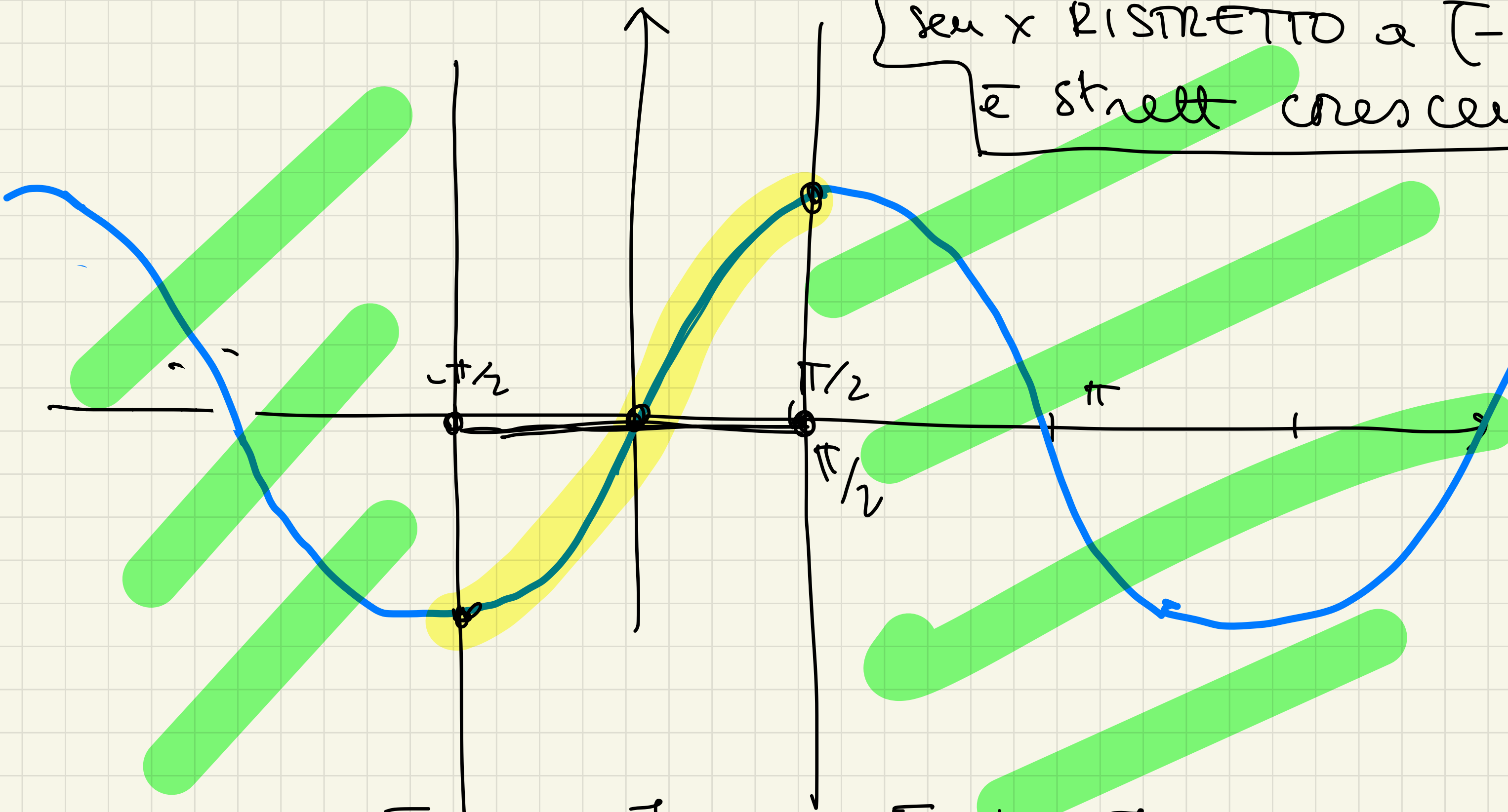
$$f(x) > 0 \quad \text{se} \quad x \in [1, +\infty)$$

$$f(x) < 0 \quad \text{se} \quad x \in (-\infty, -1]$$

Funzione sen x

RESTRINGO a $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\boxed{\text{sen } x \text{ RISTRETTO a } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ è strett crescente}}$



$$\text{sen } x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

$\text{sen}\,x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$

$\text{sen}\,0 = 0 \quad \text{sen}\,\frac{\pi}{2} = 1 \quad \text{sen}\left(-\frac{\pi}{2}\right) = -1$

$(\text{sen}(-x) = -\text{sen}\,x)$

è monotone crescente → ha INVERSA

FUNZIONE INVERSA

$\text{arcsin}\,x : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

arcoseno di x

DOMINIO è $[-1, 1]$

$\text{arcsen}(\text{sen}\,x) = x \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\text{sen}(\text{arcsen}\,x) = x \quad \forall x \in [-1, 1]$

$\arcsin(\sin x) = x \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin(\arcsin x) = x \quad \forall x \in [-1, 1]$

$\arcsin x : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

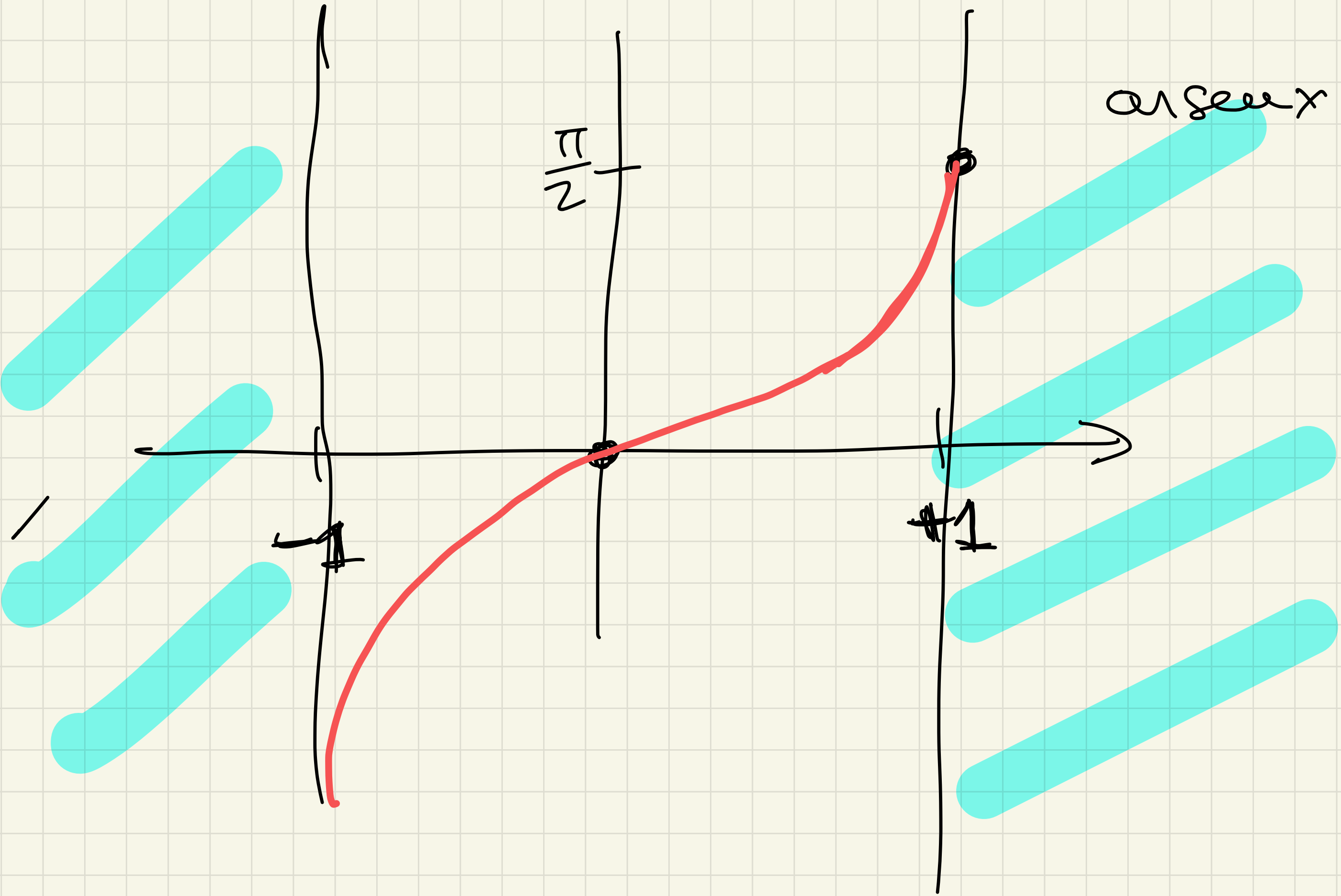
DOMINIO $[-1, 1]$, LIMITATA $-\frac{\pi}{2} \leq \arcsin x \leq +\frac{\pi}{2}$

monotona crescente e DISPARI $\boxed{\arcsin(-x) = -\arcsin x}$

$\arcsin 0 = 0$

$\arcsin 1 = \frac{\pi}{2} \qquad \arcsin(-1) = -\frac{\pi}{2}$

$\arcsin x \geq 0 \iff x \geq 0 \quad (x \leq 1)!$


ailleurs
$\frac{\pi}{2}$
-1
$+1$

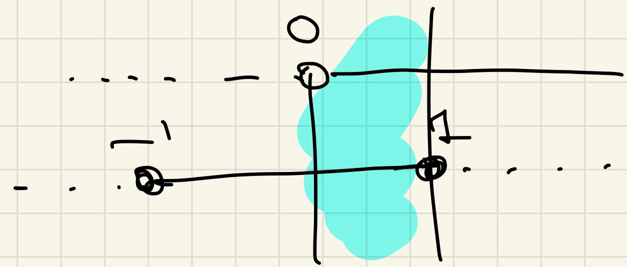
Es $\boxed{f(x) = \lg x - \arcsin x}$

dominio, segno, simmetrie

$\lg x$ è ben definito se $\boxed{x > 0}$

$\arcsin x$ è ben definito se $\boxed{-1 \le x \le 1}$

$$\begin{cases} x > 0 \\ -1 \le x \le 1 \end{cases}$$



DOMINIO

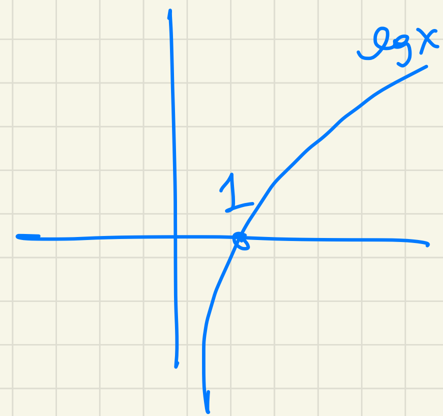
$0 < x \le 1$

$(0, 1]$

non ho simmetrie (perché dominio non è simmetrico).

<u>segno</u> $f(x) = \log x - \arcsin x$

$x \in (0, 1]$ perché $x \in D$



se $x \in (0, 1]$ $\log x \leq 0$

& $x \in (0, 1]$ $\arcsin x > 0$

$$\boxed{f(x) < 0 \quad \forall x \in D}$$

Es $\quad f(x) = \arcsin\left(\frac{1}{|x-2|}\right)$

D., segno, simmetrie

$\cancel{-1 \leq} \frac{1}{|x-2|} \leq 1$

$\begin{cases} \frac{1}{|x-2|} \leq 1 \\ \frac{1}{|x-2|} \geq -1 \end{cases}$ sempre vera

$\frac{1}{|x-2|} > 0 \quad \forall x$ sicuramente ≥ -1

$\frac{1}{|x-2|} \leq 1 \qquad x \neq 2$

$x \neq 2$

$$\frac{1}{|x-2|} \leq 1$$

$$\frac{1}{|x-2|} - 1 \leq 0$$

$$\frac{1 - |x-2|}{|x-2|} \leq 0$$

+

$N \quad 1-|x-2|$

$D: \quad |x-2| > 0$

$\Downarrow$

$1-|x-2| \leq 0$

$x \neq 2$

$$\begin{cases} 1 - |x-2| \leq 0 \\ x \neq 2 \end{cases}$$

$-|x-2| \leq -1$

$|x-2| \geq 1$

$|x-2| \geq 1$

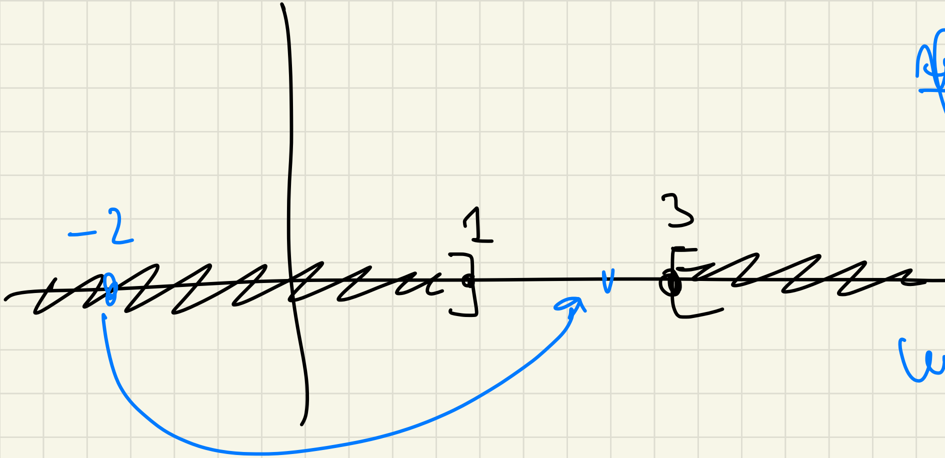
$x-2 \geq 1$ $\quad x \geq 3$

$x-2 \leq -1$ $\quad x \leq 1$

$|A| \geq B$

$A \geq B$ oppure $A \leq -B$

DOMINIO $\quad (-\infty, 1] \cup [3, +\infty)$



f non è simmetrica perché dominio non è simmetrico

$$\arcsin\left(\frac{1}{|x-2|}\right)$$

segno $\underline{f > 0}$

$$\frac{1}{|x-2|} > 0 \quad \underline{\underline{\forall x \in D}}$$

Es $\quad f(x) = \dfrac{\lg(\cos x)}{(\operatorname{sen} x)^2}$

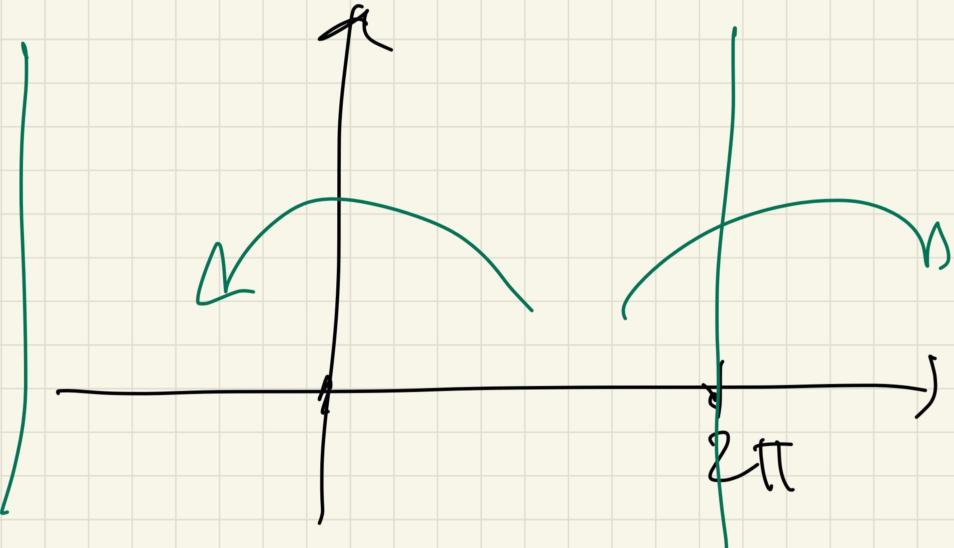
D, segno, simmetrie

$\cos(x+2\pi) = \cos x \qquad \operatorname{sen}(x+2\pi) = \operatorname{sen} x$

$f(x+2\pi) = \dfrac{\lg(\cos(x+2\pi))}{(\operatorname{sen}(x+2\pi))^2} = f(x)$

$\boxed{f \text{ è PERIODICA di periodo } 2\pi}$

Studio $f(x)$ $\boxed{\text{per } x \in [0, 2\pi]}$



$$f(x) = \frac{\lg(\cos x)}{(\operatorname{sen} x)^2}$$

DOMINIO

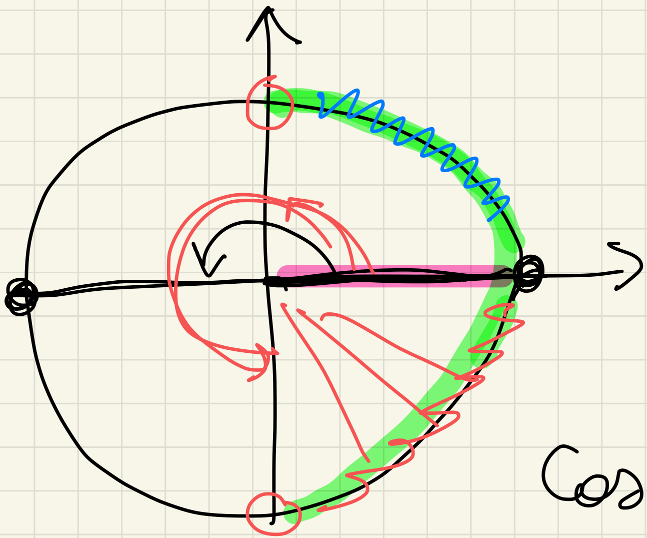
$$\cos x > 0$$

$$\operatorname{sen} x \neq 0$$

$\cos x > 0$

$\boxed{\operatorname{sen} x \neq 0}$

$x \in [0, 2\pi]$



$\operatorname{sen} 0 = \operatorname{sen} 2\pi = \operatorname{sen} \pi = 0$

$x \neq 0, \pi, 2\pi$

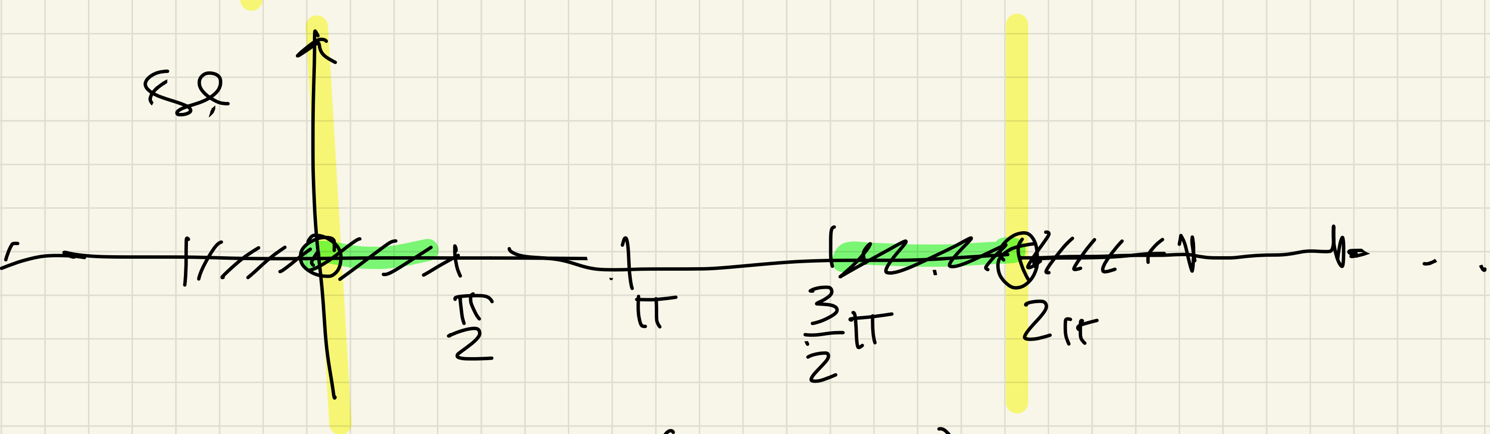
$\cos x > 0 \iff$

$x \in \left[0, \frac{\pi}{2}\right)$

$x \in \left(\frac{3}{2}\pi, 2\pi\right]$

D IN $[0, 2\pi]$

$$\boxed{x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3}{2}\pi, 2\pi\right)} + 2k\pi$$



$$f(-x) = \frac{\lg(\cos(-x))}{[\operatorname{sen}(-x)]^2} = \frac{\lg(\cos x)}{(-\operatorname{sen} x)^2} = \frac{\lg(\cos x)}{(\operatorname{sen} x)^2} = f(x)$$

f è PARI

segno $f(x) \geq 0$

$\boxed{\text{f è sempre NEGATIVA nel suo DOMINIO}}$

$$\frac{\lg(\cos x)}{(\operatorname{sen} x)^2} \geq 0 \Longrightarrow \lg(\cos x) \geq 0$$

$$\boxed{(\operatorname{sen} x)^2 > 0 \quad \forall x \in D}$$

$$\lg(\cos x) \geq 0 = \lg 1 \Longrightarrow \cos x \geq 1$$

$$x = 0 \quad x \notin D$$