By Folland $\quad f \geq 0 \quad f : \mathbb{R}^n \to [0, +\infty]$ measurable

$A = \{ (x,y) \quad x \in \mathbb{R}^n , \quad y \leq f(x) \}$ is measurable in $\mathbb{R}^{n+1}$

$$F : \underset{\mathbb{R}^{n+1}}{(x,y)} \to \underset{\mathbb{R}}{f(x) - y}$$

$$\left(\underbrace{(x,y)}_{\text{meas.}} \xrightarrow{f \times Id} (f(x), y) \xrightarrow{\text{Borel}} f(x) - y \right) \qquad \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$$

(it is measurable since composition of meas and Borel)

$$\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^2$$

$$f \times id : (x,y) \mapsto (f(x), y) \quad \text{measurable}$$

$$(\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}))$$

By Evans - Gariepy

<u>Lemma 1</u> $f : \mathbb{R}^n \to [0,+\infty)$ measurable

$$A = \{ (x,y) \quad x \in \mathbb{R}^n, \quad 0 \le y \le f(x) \} \text{ subgraph.}$$

A is measurable in $\mathbb{R}^{n+1}$.

<u>proof</u> $C_{jk} = \{ x \mid \frac{j}{k} \le f(x) < \frac{j+1}{k} \}$ $\quad j \in \mathbb{N}_0 \quad j = 0, 1, \dots$ $\quad k \in \mathbb{N} \quad k = 1, 2 \dots$

$$\bigcup_{j=0}^{\infty} C_{jk} = \{ x \mid f(x) < +\infty \} \qquad B = \{ x \quad f(x) = +\infty \}$$

$$D_k := \bigcup_{j=0}^{+\infty} \left(C_{jk} \times [0, \tfrac{j}{k}] \right) \cup B \times [0 + \infty] \to \text{measurable in } \mathbb{R}^{n+1}$$

$$E_k := \bigcup_{j=0}^{+\infty} \left(C_{jk} \times [0, \tfrac{j+1}{k}] \right) \cup B \times [0 + \infty] \to \text{meas. in } \mathbb{R}^{n+1}$$

$$\boxed{D_k \subset A \subset E_k} \qquad D = \bigcup_k D_k \qquad E = \bigcap_k E_k$$

$$D \subset A \subset E \qquad D, E \text{ measurable in } \mathbb{R}^{n+1}$$

$$|(E \setminus D) \cap B(0,R)| \le |(E_k \setminus D_k) \cap B(0,R)| \le \frac{1}{k} |B(0,R)| \longrightarrow 0$$

$k \to +\infty$

$$|(E \setminus D) \cap B(0,R)| = 0 \quad \forall R > 0$$

$$\rightsquigarrow |E \setminus D| = 0 \implies (A \setminus D) \subseteq (E \setminus D) \text{ so } |A \setminus D| = 0$$

$\boxed{A \setminus D \text{ is measurable}}$

$$A = (A \setminus D) \cup D \text{ is measurable.}$$