Theorem in $\mathbb{R}^n$, $\mathcal{H}^n = \mathcal{L}$ (n-dimensional Hausdorff m. coincides with Lebesgue)

Obs $\mathcal{H}^s$ $s<n$ in $\mathbb{R}^n$ is never a Radon measure

$K \subset\subset \mathbb{R}^n$ $|K|>0 \Rightarrow \mathcal{H}^n(K)>0$

$\Rightarrow \mathcal{H}^s(K) = +\infty \quad \forall\, s<n.$

Obs if $B \in \mathcal{B}(\mathbb{R}^n)$ $0 < |B| < +\infty$ $\mathcal{H}^n(B)$

$$\Rightarrow \mathcal{H}^s(B) = 0 \quad \forall s>n \qquad |B| = \mathcal{H}^n(B) < +\infty$$

$$\mathbb{R}^n = \bigcup_{i=1} C_i \qquad |C_i| < +\infty \qquad C_i \cap C_j = \emptyset$$

$$\Rightarrow \mathcal{H}^s(\mathbb{R}^n) = 0 \quad \forall s>n \qquad \mathcal{H}^s(\mathbb{R}^n) = \sum_i \mathcal{H}^s(C_i)$$

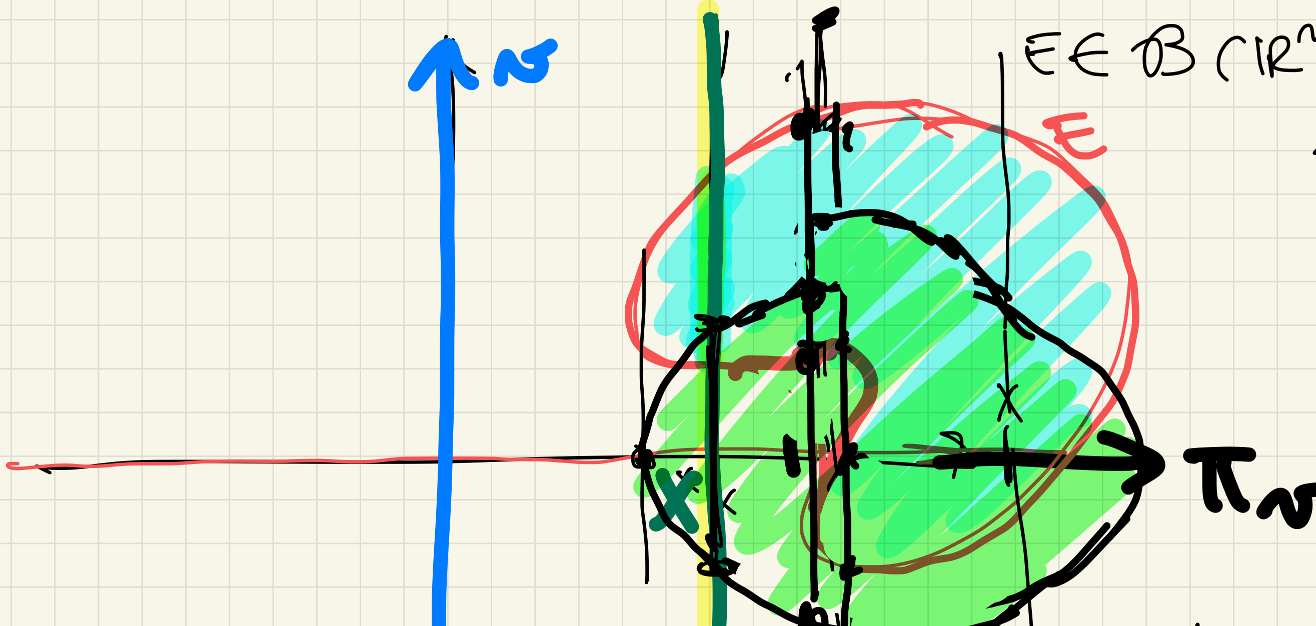
Preliminary results

STEINER SYMMETRIZATION (Steiner ~ 1840)

$\mathbb{R}^n$ $\quad$ v vector in $\mathbb{R}^n$ $\quad$ $|v| = 1$

Π_v = hyperplane in $\mathbb{R}^n$ orthogonal to v (and passing through the origin)

$E \in \mathcal{B}(\mathbb{R}^n)$ $\quad$ E Borel

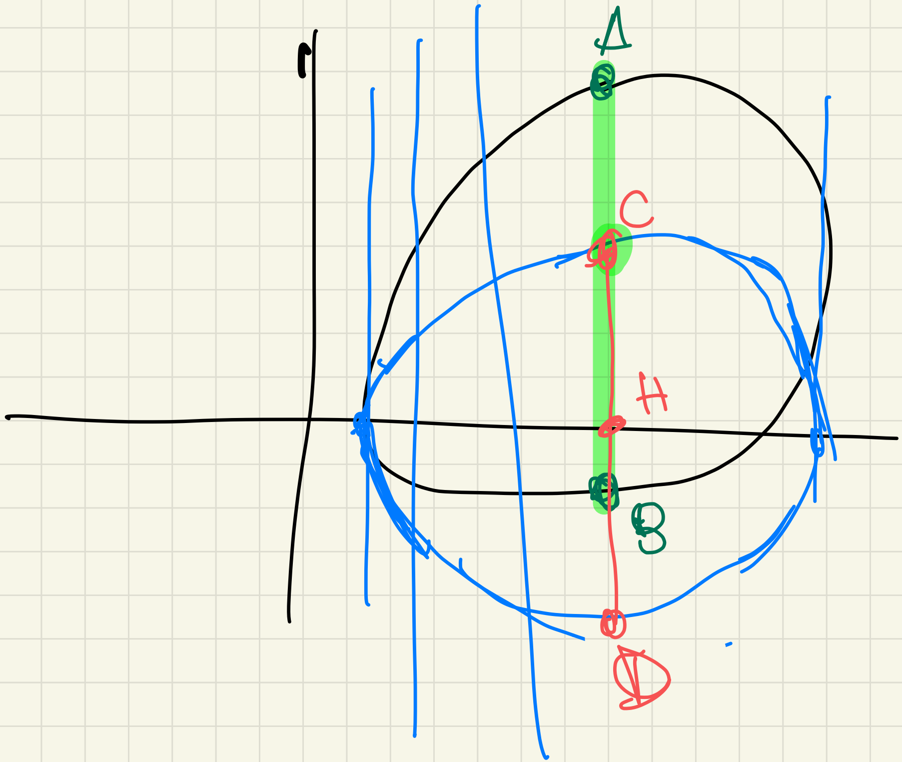


$E \in \mathcal{B}(\mathbb{R}^n)$

$\forall\, x \in \pi_v \quad t \to \underbrace{x + tv} \qquad t \in \mathbb{R}$

$L_{x,v} = \{ x + tv \,,\ t \in \mathbb{R} \}$

$$S_v(E) = \bigcup_{\substack{x \in \pi_v \\ L_{x,v} \cap E \neq \emptyset}} \left\{ x + r v \quad \boxed{|r| \leq \frac{1}{2} \mathcal{H}^1(E \cap L_{x,v})} \right.$$



$\overline{AB} = \overline{CD}$

$\overline{CH} = \overline{HD}$

$S_v(E)$ is symmetric by construction w.r.to π_v

① $|S_v(E)| = |E|$ (by Fubini)

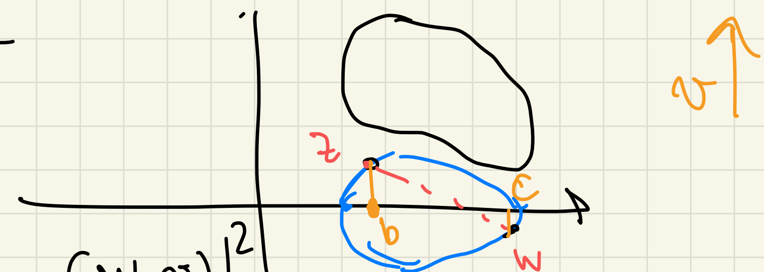
---

② $\operatorname{diam} S_v(E) \le \operatorname{diam}(E)$

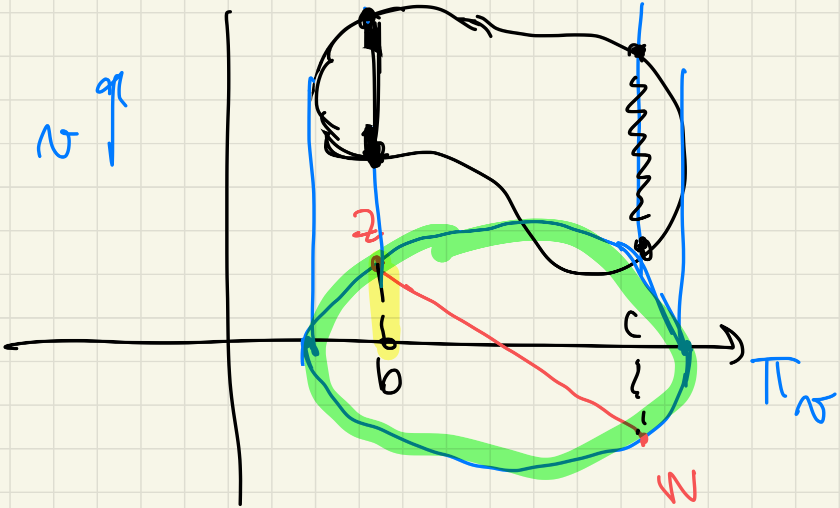
$z, w \in S_v(E)$ $\quad |z-w| = \operatorname{diam} S_v(E)$

$b := z - \underbrace{(z\cdot v)}_{\text{scalar product}} v \qquad c := w - (w\cdot v)\, v$

$b \in \pi_v \quad c \in \pi_v$



$|z-w|^2 = |b-c|^2 + |(z\cdot v) - (w\cdot v)|^2$



$|z-w| \cong \operatorname{diam} S_v(E)$

$\underline{b+tv}$ $\qquad$ $c+tv$

$s = \inf\{t \mid b+tv \in E\}$

$s' = \sup\{t \mid b+tv \in E\}$

$r = \inf\{t \mid c+tv \in E\}$

$r' = \sup\{t \mid c+tv \in E$

$$\boxed{\begin{array}{c} s'-r \\ \text{VI} \\ r'-s \end{array}}$$

$$s'-r \geq \frac{1}{2}(s'-r)+\frac{1}{2}(r'-s) = \frac{1}{2}(s'-s)+\frac{1}{2}(r'-r)$$

$$|z\cdot v - w\cdot v| \quad \wedge\!|$$

$$\geq \frac{1}{2}\mathcal{H}^1(E\cap L_{b,v}) + \frac{1}{2}\mathcal{H}^1(E\cap L_{cv}) \geq |z\cdot v| + |w\cdot v|$$

$$\operatorname{diam}^2 S_v(E) = |z-w|^2 = |b-c|^2 + |z\cdot v - w\cdot v|^2$$

$$\leq |b-c|^2 + |s' - r|^2 =$$

$$= \big|\underbrace{(b+s'v)}_{\in\, \overline{E}} - \underset{\in\, \overline{E}}{(c+rv)}\big|^2 \leq (\operatorname{diam} E)^2$$

$$\operatorname{diam} S_v(E) \leq \operatorname{diam} E .$$

Isodiametric inequality

$E \in \mathcal{B}(\mathbb{R}^n)$ $\qquad\qquad$ $\omega_n = |B(0,1)|$

$$|E| \leq \omega_n \frac{(\operatorname{diam} E)^n}{2^n}$$

$=$ if E is a ball

$n = 1$ ok..

proof $e_1 \dots e_n$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \qquad e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

$E_1 = S_{e_1}(E)$ symm. w.r. to π_{e_1}

$E_2 = S_{e_2}(E_1)$ symm. w.r.to π_{e_2}

symm. w.r.to π_{e_1}

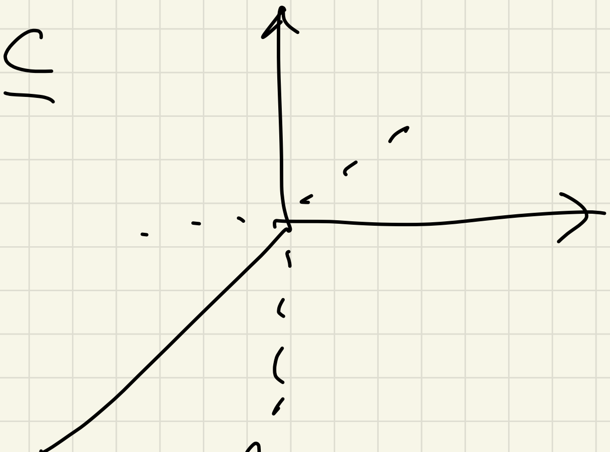
$E_n = E^* = S_{e_n}(E_{n-1})$

E^* is symmetric w.r.to $\pi_{e_1}\ \pi_{e_2}\ \pi_{e_3} \dots\ \pi_{e_n}$

$$\mathrm{diam}(E^*) \le \mathrm{diam}(E) \qquad |E^*| = |E|$$

$$E^* \subseteq B\left(0, \frac{\mathrm{diam}\, E^*}{2}\right) \subseteq$$

$$\subseteq \underbrace{B\left(0, \frac{\mathrm{diam}\, E}{2}\right)}$$



$$|E^*| = |E| \le \omega_n \left(\frac{\mathrm{diam}\, E}{2}\right)^n \qquad \square .$$

Proof theorem $\mathcal{H}^n = \mathcal{L}$

① $\forall E$ Borel $|E| \leq \mathcal{H}^n_\delta(E) \leq \mathcal{H}^n(E)$

$\forall \delta > 0$

$\delta > 0$ $E \subseteq \bigcup_i E_i$ $\operatorname{diam} E_i \leq \delta$

$$|E| \leq \sum_i |E_i| \underset{\text{isodiamet.}}{\leq} \sum_i \frac{\omega_n}{2^n} (\operatorname{diam} E_i)^n$$

taking the infimum

$$|E| \leq \mathcal{H}^n_\delta(E).$$