HAUSDORFF MEASURES

(Evans – Gariepy chapter 3
Folland – appendix)

$\omega_n = |B(0,1)|$ = volume of the n-dimensional ball of radius 1 $= \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$

$k \geq 0$

$$\omega_s := \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)} > 0 \qquad \omega_0 = 1$$

$A \subseteq \mathbb{R}^n$

$\operatorname{diam} A = \sup\{ |x-y| \mid x,y \in A \}$

$$\boxed{\frac{\omega_s}{2^s} = \left(\frac{\sqrt{\pi}}{2}\right)^s \frac{1}{\Gamma(\frac{s}{2}+1)}}$$

$s \geq 0$, $\delta > 0$

we define for $E \subseteq \mathbb{R}^n$

$$\mathcal{H}^s_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} \frac{\omega_s (\operatorname{diam} E_i)^s}{2^s} \;\middle|\; E_i \subseteq \mathbb{R}^n,\ \operatorname{diam} E_i \leq \delta,\ E \subseteq \bigcup_i E_i \right\}$$

(we may reduce to consider just E_i all open, or all closed or all convex.

$\delta_1 > \delta_2 > 0$ $E \subseteq \mathbb{R}^n$ $\boxed{\mathcal{H}^s_{\delta_1}(E) \leq \mathcal{H}^s_{\delta_2}(E)}$

δ fixed

$\operatorname{diam} E_i \leq \delta_2 < \delta_1$ $\downarrow$

$$\underbrace{\mathcal{H}^s(E)}_{\text{s-dimensional Hausdorff measure of } E.} := \underbrace{\sup_{\delta>0} \mathcal{H}^s_\delta(E)} = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(E)$$

<u>Remark</u> (1) $\forall\, s \geq 0,\ \delta > 0$

$\underset{\subseteq \mathbb{R}^n}{E} \longrightarrow \underset{\mathcal{H}^s(E)}{\mathcal{H}^s_\delta(E)}$ is an OUTER MEASURE,

$A, B \subseteq \mathbb{R}^n$ $\operatorname{dist}(A,B) \geq 2\delta$ "$\mathcal{H}^s_\delta(A \cup B) = \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$"

$\Rightarrow$ letting $\delta \to 0^+$

① $\text{dist}(A,B) > 0$ $\qquad \mathcal{H}^S(A \cup B) = \mathcal{H}^S(A) + \mathcal{H}^S(B)$

② $\mathcal{H}^S(\cdot)$ is an outer measure

$\downarrow$ $\mathcal{M} = \{$ measurable sets for $\mathcal{H}^S$: A is measurable if $\forall B \subseteq \mathbb{R}^n$ $\quad \mathcal{H}^S(B) = \mathcal{H}^S(A \cap B) + \mathcal{H}^S(B \setminus A)\}$

$\mathcal{H}^S|_{\mathcal{M}}$ is a measure (by Caratheodory criterium) is a BORELLIAN MEASURE

by prop ① $\mathcal{B} \subseteq \mathcal{M}$ (closed sets are measurables w.r.t. $\mathcal{H}^S \Rightarrow$ all borel sets are measurable w.r.t $\mathcal{H}^S$).

Obs. (by def). $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$

$\lambda > 0 \quad x \in \mathbb{R}$

$\# E_i$

Obs $s=0$ $\quad \mathcal{H}^0_\delta(E) = \inf \left\{ \overbrace{\frac{1}{2^0} \underline{\sum_i (\operatorname{diam} E_i)^0}} \quad \begin{array}{l} \bigcup_i E_i \supseteq E \\ \operatorname{diam} E_i \leq \delta \end{array} \right\}$

$\mathcal{H}^0$ is the COUNTING MEASURE

$$\mathcal{H}^0(E) = \#\{x \in \mathbb{R}^n,\ x \in E\}$$

Obs $f: \mathbb{R}^n \to \mathbb{R}^n$ a Lipschitz function with Lipschitz constant $L > 0$ $\quad |f(x) - f(y)| \leq L|x-y|$

$\forall x, y \in \mathbb{R}^n$.

$$\mathcal{H}^s(f(E)) \leq L^s \mathcal{H}^s(E)$$

Observation $\boxed{s_1 < s_2}$ δ fixed $E \subseteq \mathbb{R}^n$ fixed

$s_2 - s_1 > 0$

$$\mathcal{H}^{s_2}_{\delta}(E) \leq \delta^{s_2-s_1} \underline{c(s_2,s_1)} \mathcal{H}^{s_1}_{\delta}(E)$$

let us fix E_i $\operatorname{diam} E_i \leq \delta$ $E \subseteq \bigcup_i E_i$

$$\mathcal{H}^{s_2}_{\delta}(E) \leq \frac{\omega_{s_2}}{2^{s_2}} \sum_i (\operatorname{diam} E_i)^{s_2} = \frac{\omega_{s_2}}{2^{s_2}} \sum_i (\operatorname{diam}(E_i))^{s_1} \cdot \underbrace{(\operatorname{diam} E_i)}_{\leq \delta}^{s_2-s_1}$$

$$= \frac{\omega_{s_2}}{2^{s_2}} \delta^{s_2-s_1} \underline{\sum_i (\operatorname{diam}(E_i))^{s_1}} =$$

$$= \left(\frac{\omega_{s_2}}{2^{s_1}} \cdot \frac{2^{s_1}}{\omega_{s_1}}\right) \delta^{s_2-s_1} \left[\frac{\omega_{s_1}}{2^{s_1}} \sum_i (\operatorname{diam} E_i)^{s_1}\right]$$

$$= \underbrace{\left(\frac{\sqrt{\pi}}{2}\right)}_{s_1}^{s_2-s_1} \frac{\Gamma\left(\frac{s_1}{2}+1\right)}{\Gamma\left(\frac{s_2}{2}+1\right)} \cdot \delta^{s_2-s_1} \left[\sum_i \frac{\omega_{s_1}}{2^{s_1}} (\operatorname{diam} E_i)^{s_1}\right]$$

$$\mathcal{H}^{s_2}_{\delta}(E) \leq \frac{\Gamma\left(\frac{s_1}{2}+1\right)}{\Gamma\left(\frac{s_2}{2}+1\right)} \delta^{s_2-s_1} \left[\sum_i \frac{\omega_{s_1}}{2^{s_1}} (\operatorname{diam} E_i)^{s_1}\right]$$

$$\forall E_i \quad \operatorname{diam} E_i \leq \delta \qquad \bigcup_i E_i \supseteq E$$

$\Rightarrow$ taking the infimum

$$\mathcal{H}^{s_2}_{\delta}(E) \leq \frac{\Gamma\left(\frac{s_1}{2}+1\right)}{\Gamma\left(\frac{s_2}{2}+1\right)} \delta^{s_2-s_1} \quad \mathcal{H}^{s_1}_{\delta}(E)$$

$\boxed{\text{letting } \delta \to 0}$

$$0 \le \mathcal{H}^{s_2}_{\delta}(E) \le \frac{\Gamma\left(\frac{s_1}{2}+1\right)}{\Gamma\left(\frac{s_2}{2}+1\right)} \underbrace{\delta^{s_2-s_1}}_{\downarrow\, 0} \mathcal{H}^{s_1}_{\delta}(E)$$

$s_2 > s_1$

$\mathcal{H}^{s}(E) = \lim_{\delta \to 0^+} \mathcal{H}^{s}_{\delta}(E$

$\delta^{s_2-s_1} \to 0$ as $\delta \to 0$

if $\mathcal{H}^{s_2}(E) > 0 \implies \mathcal{H}^{s_1}(E) = +\infty$

if $\mathcal{H}^{s_1}(E) < +\infty \implies \mathcal{H}^{s_2}(E) = 0$

If for some s $\mathcal{H}^{s}(E) > 0$ (and $\neq +\infty$)
$\implies \mathcal{H}^{k}(E) = +\infty \ \forall k < s$, $\mathcal{H}^{k}(E) = 0 \ \forall k > s$

$\Rightarrow$ there exists at most one $s \geq 0$ such that $\mathcal{H}^s(E) \in (0, +\infty)$

$\mathcal{H}^k(E) = +\infty \qquad s \qquad \mathcal{H}^k(E) = 0$

Hausdorff dim of $E \subseteq \mathbb{R}^n$

$$\dim_{\mathcal{H}}(E) = \inf \{ s > 0 \ \ \mathcal{H}^s(E) = 0 \}$$
$$= \sup \{ s > 0 \ \ \mathcal{H}^s(E) = +\infty \}$$

Theorem If $s = n$ $\mathcal{H}^n(E) = |E| \quad \forall E \in \mathcal{B}$.

(n-dim Hausdorff measure in $\mathbb{R}^n$ is LEBESGUE)

proof tomorrow

Conseq.: if $E \subseteq \mathbb{R}^n$ (Borel) ~~bdd~~ $|E| < +\infty$ $\rightarrow$ $\boxed{\mathcal{H}^s(E) = 0}$ $\boxed{\forall s > n}$.

Conseq. $\mathcal{H}^s$ for $s < n$ is NOT A RADON MEASURE

$K \subseteq \mathbb{R}^n$ K compact $|K| > 0 \Rightarrow \mathcal{H}^n(K) > 0$

$\Rightarrow \mathcal{H}^s(K) = +\infty \quad \forall s < n$.

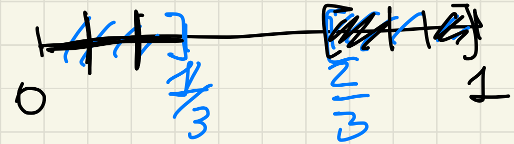
A Borel set of positive Lebesgue measure has Hausdorff dimension $= n$.

A Borel set of NULL LEBESGUE measure has Hausdorff dim $\leq n$

$A = \{x_0\}$ $\qquad \mathcal{H}^0(A) = 1$

$\mathcal{H}^s(A) = 0 \quad \forall\ s > 0$

Ex **Cantor set**



$C_0 = [0,1]$ $\qquad C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$

$$C_2 = [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1]$$

$$C_{n+1} = \frac{1}{3} C_n \cup \left(\frac{2}{3} + \frac{1}{3} C_n\right)$$

$$C = \bigcap_n C_n$$ CLOSED

$C = \bigcap C_n$

$C = \frac{1}{3} C \cup \left(\frac{2}{3} + \frac{1}{3} C\right)$

$\downarrow$

$\mathcal{H}^s(C) = \mathcal{H}^s\left(\frac{1}{3} C \cup \left(\frac{2}{3} + \frac{1}{3} C\right)\right)$

$= 2\, \mathcal{H}^s\left(\frac{1}{3} C\right)$

$= 2 \cdot \left(\frac{1}{3}\right)^s \mathcal{H}^s(C)$

$\boxed{\mathcal{H}^s(C) = \frac{2}{3^s} \mathcal{H}^s(C)}$

$\frac{2}{3^s} = 1$

$2 = 3^s$

$\boxed{s = \log_3 2} < 1$

(for $s \neq \log_3 2$ NECESSARILY $H^s(C) = +\infty$ o 0)

Cantor set has Lebesgue measure $= 0$
$\subseteq \mathbb{R}$

has Hausdorff dimension $\log_3 2$