COMPUTABILITY (15/10/2024)

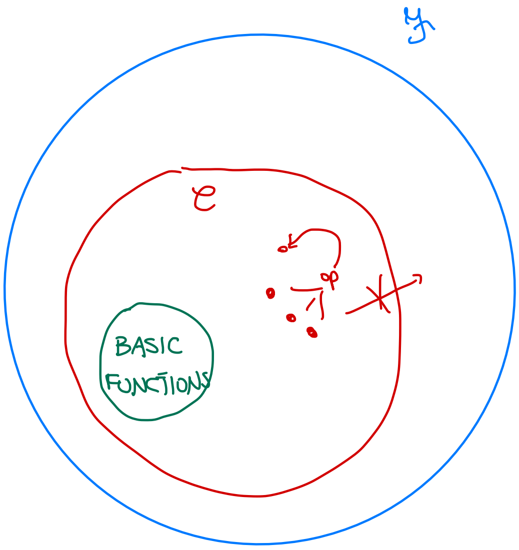
* Generation of computable functions

$\mathcal{C}$ closed under

→ composition

→ primitive recursion

→ unbounded minimalisation



* BASIC FUNCTIONS

① constant zero $z: \mathbb{N}^k \to \mathbb{N}$ $z(\vec{x}) = 0 \quad \forall \vec{x} = (x_1, \dots, x_k) \in \mathbb{N}^k$

② successor $s: \mathbb{N} \to \mathbb{N}$ $s(x) = x + 1$

③ projection $U_j^k : \mathbb{N}^k \to \mathbb{N}$ $U_j^k(\vec{x}) = x_j$

They are in $\mathcal{C}$ since they are computed by

① $Z(1)$

② $S(1)$

③ $T(j, 1)$

* Notation

given a program P

- $\rho(P) = \max \{ n \mid$ register R_n is referred in P $\}$
- $\ell(P) =$ length of P
- P is in standard form if whenever it terminates it does at line $\ell(P)+1$

concatenation of P, Q program

P this means P
Q ⤳ Q' ← update of Q replacing $J(m, n, t)$
→ $J(m, n, t + \ell(P))$

- given P program, we write

$P[i_1 \dots i_k \to i]$

program taking the input from $R_{i_1}, \dots, R_{i_k}$ and puts the output in R_i without assuming that the memory different from input is $\emptyset$

$$(*) \left\{ \begin{array}{l} T(i_1, 1) \\ \vdots \\ T(i_k, k) \\ Z(k+1) \\ \vdots \\ Z(\rho(P)) \\ P \\ T(1, i) \end{array} \right.$$

problem $P[2, 1 \to 1]$

what I expect

x	y	...

$\rightsquigarrow$

y	x	0	...

what I get

$T(2,1)$
$T(1,2)$
P

y	y	...

EXERCISE: solve the issue with (*)

* COMPOSITION

Given $f : \mathbb{N}^k \to \mathbb{N}$, $g_1, \dots, g_k : \mathbb{N}^m \to \mathbb{N}$

define $h : \mathbb{N}^m \to \mathbb{N}$ for $\vec{x} \in \mathbb{N}^m$

$$h(\vec{x}) = \begin{cases} f(g_1(\vec{x}), \dots, g_k(\vec{x})) & \text{if } g_1(\vec{x})\downarrow, \dots, g_k(\vec{x})\downarrow \text{ and } f(g_1(\vec{x}), \dots, g_k(\vec{x}))\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Eg. $z(x) = 0 \quad \forall x$

$\emptyset(x)\uparrow \quad \forall x \qquad z(\emptyset(x)) \uparrow \quad \forall x$

$U_1^2(x,y) = x \quad \forall (x,y) \qquad U_1^2(x, \emptyset(y)) \uparrow \quad \forall x, y$

$\neq x$

Proposition: $\mathcal{C}$ is closed under (generalized) composition

proof Given $f : \mathbb{N}^k \to \mathbb{N}$ $\quad g_1, \dots, g_k : \mathbb{N}^k \to \mathbb{N}$ in $\mathcal{C}$

and consider

$$h : \mathbb{N}^n \to \mathbb{N} \qquad h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x}))$$

We want to "construct" a program for h using the programs for f $\; g_1 \dots g_k$

call such programs $\quad F \quad G_1 \dots G_k$

1		n		m	m+1		m+n	m+n+1		m+n+k
x_1	...	x_n			x_1	...	x_n	$g_1(\vec{x})$		$g_k(\vec{x})$

$$m = \max\{\rho(F), \rho(G_1), \dots, \rho(G_k), k, n\}$$

$T(1, m+1)$
$\vdots$
$T(n, m+n)$

$G_1[m+1 \dots m+n \to m+n+1]$
$\vdots$
$G_k[m+1 \dots m+n \to m+n+k]$

$F[m+n+1 \dots m+n+k \to 1]$

program computing h
$\downarrow$
$h \in \mathcal{C}$

$\square$

Example $f : \mathbb{N}^2 \to \mathbb{N}$

$f(x_1, x_2) = x_1 + x_2$ $\quad \in \mathcal{C}$

we want to derive

$$g : \mathbb{N}^3 \to \mathbb{N} \qquad g(x_1, x_2, x_3) = x_1 + x_2 + x_3 \qquad \in \mathcal{C}$$

in fact $\quad g(x_1, x_2, x_3) = f(f(x_1, x_2), x_3)$ $\qquad \vec{x} = (x_1, x_2, x_3)$

$$\underbrace{f\Big(\underbrace{\underbrace{f}_{\mathbb{N}^2\to\mathbb{N}}\big(\underbrace{U^3_1(\vec{x})}_{\mathbb{N}^3\to\mathbb{N}}\ \underbrace{U^3_2(\vec{x})}_{\mathbb{N}^3\to\mathbb{N}}\big)}_{\mathbb{N}^3\to\mathbb{N}},\ \underbrace{U^3_3(\vec{x})}_{\mathbb{N}^3\to\mathbb{N}}\Big)}_{\mathbb{N}^3\to\mathbb{N}}$$

$\mathbb{N}^2 \to \mathbb{N}$

* <u>Example</u> : Let $f : \mathbb{N} \to \mathbb{N}$ be computable & TOTAL and consider

$$Q_f(x,y) \equiv \text{“ } f(x) = y \text{ ”}$$ is this decidable ?

$$\chi_{Q_f}(x,y) = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{otherwise} \end{cases}$$ computable

We know that

$$\chi_{Eq}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\chi_{Q_f}(x,y) = \chi_{Eq}(f(x), y)$$ computable by composition

* <u>Primitive recursion</u>

$$\begin{cases} 0! = 1 \\ (n+1)! = (n+1) * n! \end{cases}$$

$$\begin{cases} fib(0) = 1 \\ fib(1) = 1 \\ fib(n+2) = fib(n) + fib(n+1) \end{cases}$$

<u>Def</u>. Given $f : \mathbb{N}^k \to \mathbb{N}$

$g : \mathbb{N}^{k+2} \to \mathbb{N}$

define $h : \mathbb{N}^{k+1} \to \mathbb{N}$
$$\left(\begin{cases} h(\vec{x}, 0) = f(\vec{x}) \\ h(\vec{x}, y+1) = g(\vec{x}, y, h(\vec{x}, y)) \end{cases} \right)$$

take x s.t. $x = \dfrac{\sqrt{x} \cdot \log x}{e^x}$ → is there a solution ?

→ is it unique ?

($(\mathcal{F}, \leq)$ complete partial order
+ continuous operator } → existence of a solution (least))

uniqueness follows by induction

Example

→ $h : \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x+y$

$$\begin{cases} x+0 = x \\ x+(y+1) = (x+y)+1 \end{cases} \qquad \begin{aligned} f(x) &= x \\ g(x,y,z) &= z+1 \end{aligned}$$

→ $h : \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x * y$

$$\begin{cases} x * 0 = 0 \\ x * (y+1) = (x * y) + x \end{cases} \qquad \begin{aligned} f(x) &= 0 \\ g(x,y,z) &= z + x \end{aligned}$$

Proposition : $\mathcal{C}$ is closed by primitive recursion

proof Let $f : \mathbb{N}^k \to \mathbb{N}$, $g : \mathbb{N}^{k+2} \to \mathbb{N}$ be in $\mathcal{C}$

and let F, G programs for f, g (std form).

Define $h : \mathbb{N}^{k+1} \to \mathbb{N}$

$$h(\vec{x},0) = f(\vec{x})$$

$$h(\vec{x},y+1) = g(\vec{x},y,h(\vec{x},y))$$

Idea : for computing $h(\vec{x},y)$

$h(\vec{x},0) = f(\vec{x})$ use F

$h(\vec{x},1) = g(\vec{x},0,h(\vec{x},0))$ use G

$\vdots$

$h(\vec{x},i) = g(\vec{x},i,h(\vec{x},i-1))$

check $i = y$? yes → output $h(\vec{x},i)$

no , i++ and continue

1		k	k+1		m	m+1		m+k	m+k+1		m+k+3
x_1	....	x_k	y			x_1	...	x_k	i	$h(\vec{x},i)$	y

$m = \max\{\rho(F), \rho(G), k+2\}$

↑ m+k+2

initially 0

$T(1, m+1)$

$\vdots$

$T(k, m+k)$

$T(k+1, m+k+3)$

$F[m+1, \ldots, m+k \to m+k+2]$

LOOP: $J(m+k+1, m+k+3, RES)$ // $i = y$?

$G[m+1, \ldots, m+k+2 \to m+k+2]$ // $h(\vec{x}, i+1) = g(\vec{x}, i, h(\vec{x}, i))$

$S(m+k+1)$ // $i{+}{+}$

$J(1, 1, LOOP)$

RES: $T(m+k+2, 1)$

$\square$

Example

→ $h : \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x + y$

$$\begin{cases} x + 0 = x \\ x + (y+1) = (x+y) + 1 \end{cases}$$

$f(x) = x$

$g(x,y,z) = z + 1$

→ $h : \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x * y$

$$\begin{cases} x * 0 = 0 \\ x * (y+1) = (x*y) + x \end{cases}$$

$f(x) = 0$

$g(x,y,z) = z + x$

→ expomential x^y

$x^0 = 1$

$x^{y+1} = (x^y) * x$

→ predecessor $\quad y \dot{-} 1$

$$0 \dot{-} 1 = 0$$
$$(y+1) \dot{-} 1 = y$$

→ difference $\quad x \dot{-} y = \begin{cases} 0 & x \leq y \\ x - y & x > y \end{cases}$

$$x \dot{-} 0 = x$$
$$x \dot{-} (y+1) = \underline{(x \dot{-} y)} \dot{-} 1$$

→ sign $\quad \mathrm{sign}(y) = \begin{cases} 0 & y = 0 \\ 1 & y > 0 \end{cases}$

$$sg(0) = 0$$
$$sg(y+1) = 1$$

→ $\overline{sg}(y) = \begin{cases} 0 & \text{if } y > 0 \\ 1 & \text{if } y = 0 \end{cases}$ $\qquad$ exercise $\quad$ (SOLUTION $\overline{sg}(x) = 1 \dot{-} sg(x)$)

→ $\min(x,y) = \overbrace{x \dot{-} \underbrace{\overbrace{(x \dot{-} y)}^{0}}}^{x}$ $\qquad y \geq x$

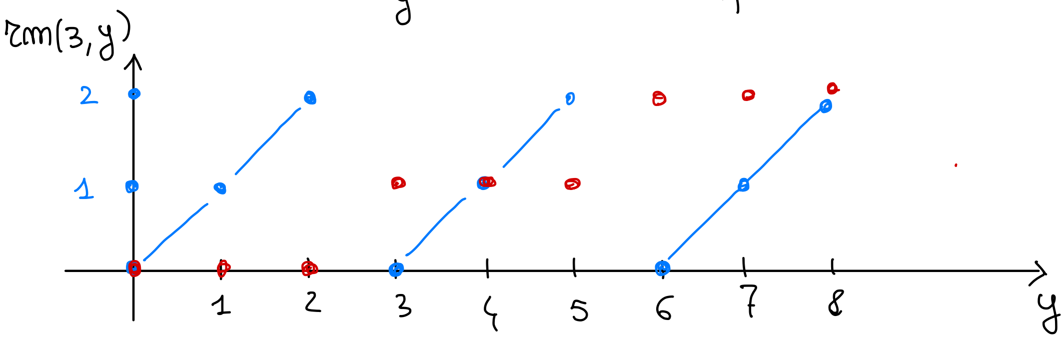
$\qquad x - (x - y) = y \qquad y < x$

→ $\max(x,y)$ $\qquad$ exercise $\quad$ (SOLUTION $\max(x,y) = x + y \dot{-} x$)

→ REMAINDER

$rm(x,y)$ = remainder of y divided by x

$$= \begin{cases} y \bmod x & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases}$$



$$rm(x, 0) = 0$$
$$rm(x, y+1) = \begin{cases} rm(x,y) + 1 & \text{if } rm(x,y) + 1 < x \\ 0 & \text{otherwise} \end{cases}$$

$$= (rm(x,y)+1) * sg(x \dot{-} (rm(x,y)+1))$$

$$\text{something} \begin{cases} 1 & rm(x,y)+1 < x \\ 0 & rm(x,y)+1 \geq x \end{cases}$$

* QUOTIENT

$qt(x,y) = y \text{ div } x$ (convention $qt(0,y) = 0$)

(exercise) SOLUTION:

$$qt(x,0) = 0$$

$$qt(x,y+1) = \begin{cases} qt(x,y)+1 & \text{if } rm(x,y+1) = 0 \\ qt(x,y) & \text{otherwise} \end{cases}$$

$$= qt(x,y) + \overline{sg}(rm(x,y+1))$$