COMPUTABILITY (14/10/2024)

EXERCISE : URM^s – machine : change URM

$T(\cancel{m}, n)$ $\qquad$ $T_s(m, n)$ $\qquad$ $r_m \leftrightarrow r_n$

$$\mathcal{C}^s = \mathcal{C}$$

proof

$(\mathcal{C} \subseteq \mathcal{C}^s)$ Given $f \in \mathcal{C}$ $\quad f : \mathbb{N}^k \to \mathbb{N}$ $\quad$ i.e. there is P URM – program such that $f_P^{(k)} = f$. Hence (by previous exercise) there is P' not using T - instructions s.t. $f_P^{(k)} = f_{P'}^{(k)}$ since P' is a program also of the URM^s machine

$$f = f_{P'}^{(k)} \in \mathcal{C}^s$$

$(\mathcal{C}^s \subseteq \mathcal{C})$ Take $f : \mathbb{N}^k \to \mathbb{N}$ $\quad f \in \mathcal{C}^s$ and let P URM^s program for f

$$f_P^{(k)} = f$$

We transform P into a program P' of URM s.t. $f_P^{(k)} = f_{P'}^{(k)}$.

$$T_s(m, n) \quad \rightsquigarrow \quad \begin{cases} T(n, i) \\ T(m, n) \\ T(i, m) \end{cases} \qquad i \text{ “not used” by } P$$

> A URM^s program P can be transformed into a URM program P' such that $f_P^{(k)} = f_{P'}^{(k)}$

We proceed by induction on h = number of T_s - instructions in P

$(h = 0)$ trivial, P is already a URM - program, take $P' = P$

$(h \to h+1)$ Let P be URM^s program with $h+1$ T_s - instructions

$$P \begin{cases} I_1 \\ \vdots \\ I_t \quad T_s(m, n) \\ \vdots \\ I_s \end{cases} \quad \rightsquigarrow \quad P'' \begin{cases} I_1 & \vdots \\ \vdots & \\ I_t & J(1, 1, SUB) \\ \vdots & \\ I_s & \vdots \\ I_{s+1} & J(1, 1, END) \\ SUB: & T(n, i) \\ & T(m, n) \\ & T(i, m) \\ & J(1, 1, t+1) \\ END: & \end{cases}$$

We need

→ P well formed (if it terminates, it does at time s+1)

→ $i = \max(\{m \mid R_m \text{ is used in } P\} \cup \{k\}) + 1$

Then $f_{P''}^{(k)} = f_P^{(k)}$ and P'' has h T_S - instructions.

Hence by ind. hyp. there is P' URM - program such that $f_{P'}^{(k)} = f_{P''}^{(k)}$

Thus

$$f_P^{(k)} = f_{P''}^{(k)} = f_{P'}^{(k)}$$

$\square$

The proof is <u>wrong</u> because I am using the inductive hyp. on P'' which is not a URM^S program (it uses both T_S and T)

<u>Solution</u> : prove a stronger result

" Every program P which uses both T and T_S instructions can be transformed into a program P' which uses only T hence a URM - program st. $f_P^{(k)} = f_{P'}^{(k)}$ "

(same proof, try!)

<u>EXERCISE</u> : Consider $URM^=$ without jump instructions

$$\mathcal{C}^= \subsetneq \mathcal{C}$$

<u>proof</u>

A $URM^=$- program

$$P \left\{ \begin{array}{l} I_1 \\ \vdots \\ I_s \end{array} \right.$$

$\ell(P) = s$ length of P

P terminates in $\ell(s)$ steps

⇒ $\mathcal{C}^=$ contains only total functions ⇝ $\mathcal{C}^= \subsetneq \mathcal{C}$

e.g. $f : \mathbb{N} \to \mathbb{N}$
$f(x)\uparrow \quad \forall x \in \mathbb{N}$

$f \in \mathcal{C}$
$f \notin \mathcal{C}^=$

$J(1,1,1)$

(saying "it uses Jumps" not sufficient
$J(1,1,2)$ computes $f(x) = x \in \mathcal{C}^=$)

restrict to unary functions

$\to$ $f(x) = x + c$ $\quad c \in \mathbb{N}$ suitable constant

or

$\to$ $f(x) = c$

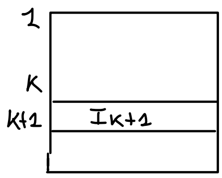
Denote $r_1(x, k)$ = content of register 1 after k steps of computation starting from $x \,|\, 0 \,|\, 0 \,|\, \dots$

I prove by induction on k that $r_1(x,k) = \begin{cases} x+c \\ c \end{cases}$

$(k=0)$ $\quad r_1(x, 0) = x + 0 \qquad c = 0 \qquad$ ok

$(k \to k+1)$ By inductive hyp. $r_1(x, k) = \begin{cases} x+c \\ c \end{cases}$

after $k+1$ steps. ...

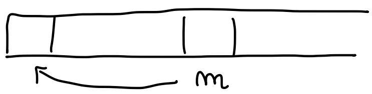


different cases according to the shape of I_{k+1}

* $I_{k+1} = Z(m)$ $\quad \begin{cases} - \; m = 1 & r_1(x, k+1) = 0 & \text{ok} \\ - \; m > 1 & r_1(x, k+1) = r_1(x,k) & \text{ok by ind. hyp.} \end{cases}$

* $I_{k+1} = S(m)$ $\quad \begin{cases} - \; m = 1 & r_1(x, k+1) = r_1(x,k)+1 & \text{ok, by ind. hyp.} \\ - \; m > 1 & r_1(x, k+1) = r_1(x,k) & \text{" " " "} \end{cases}$

* $I_{k+1} = T(m, n)$ $\quad \begin{cases} - \; n > 1 \text{ or } m = 1 & r_1(x, k+1) = r(x,k) \quad \text{ok by ind. hyp.} \\ - \; n = 1 \text{ and } m > 1 & \end{cases}$



The key observation is that the same property can be shown for all registers $r_j(x, k)$ = content of R_j after k steps of computations starting from $x \,|\, 0 \,|\, 0 \,|\, \dots$

and show by induction that for all k

$$r_j(x,k) = \begin{cases} c_j \\ x + c_j \end{cases} \qquad \text{for } c_j \text{ suitable constants}$$

[Home exercise]

for h-ary functions

$$f^{(h)}(x_1, \ldots, x_h) = \begin{cases} c \\ x_j + c \end{cases}$$

* <u>Decidable Predicate</u>

$div(x,y) =$ "x divides y"

$div \subseteq \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$

$div = \{ (m, m * k) \mid m, k \in \mathbb{N} \}$

or

$div : \mathbb{N} \times \mathbb{N} \to \{ true, false \}$

k-ary predicate $\quad Q(x_1, \ldots, x_k) \subseteq \mathbb{N}^k$

$Q : \mathbb{N}^k \to \{ \underset{1}{\underset{\uparrow}{true}}, \underset{0}{\underset{\uparrow}{false}} \}$

<u>Def</u> (decidable predicate) : Let $Q(x_1, \ldots, x_k) \subseteq \mathbb{N}^k$ k-ary predicate

Q is DECIDABLE if

$\chi_Q : \mathbb{N}^k \to \mathbb{N}$

$$\chi_Q(x_1, \ldots, x_k) = \begin{cases} 1 & \text{if } Q(x_1, \ldots, x_k) \\ 0 & \text{otherwise} \end{cases}$$

is URM-computable.

<u>Example</u> : $Q(x_1, x_2) \subseteq \mathbb{N}$

$Q(x_1, x_2) \equiv$ "$x_1 = x_2$" decidable

$$\chi_Q(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$ is computable

output ↓

x_1	x_2	0	0	...

J(1, 2, TRUE)

FALSE : J(1, 1, RES)

TRUE : S(3)

RES : T(3, 1)

Example : $Q(x) \equiv$ "x is even" decidable

1	2	3
x	0	0
	↑ K	↑ result

EVEN : $J(1,2,\text{TRUE})$
$S(2)$

ODD : $J(1,2,\text{FALSE})$
$S(2)$
$J(1,1,\text{EVEN})$

TRUE : $S(3)$

FALSE : $T(3,1)$

* Computability over other domains ?

D countable

$\alpha : D \to \mathbb{N}$ bijective "effective"
(α^{-1} effective)

A^*, $\mathbb{Q}$, $\mathbb{Z}$, ...

$\cancel{\mathbb{R}}$

Given $f : D \to D$ function is <u>computable</u> if

$$\begin{tikzcd} D \arrow[r, "f"] & D \arrow[d, "\alpha"] \\ \mathbb{N} \arrow[u, "\alpha^{-1}"] \arrow[r] & \mathbb{N} \end{tikzcd}$$

$f^* : \alpha \circ f \circ \alpha^{-1} : \mathbb{N} \to \mathbb{N}$

is URM - computable

Example : Computability on $\mathbb{Z}$

$\alpha : \mathbb{Z} \to \mathbb{N}$

$$\alpha(z) = \begin{cases} 2z & \text{if } z \geq 0 \\ -2z-1 & \text{if } z < 0 \end{cases}$$

$\mathbb{N}$	0	1	2	3	4	5	6	7
$\mathbb{Z}$	0	−1	1	−2	2	−3	3	...

$$\alpha^{-1} : \mathbb{N} \to \mathbb{Z}$$

$$\alpha^{-1}(m) = \begin{cases} m/2 & m \text{ is even} \\ -\dfrac{m+1}{2} & m \text{ is odd} \end{cases}$$

$$f : \mathbb{Z} \to \mathbb{Z}$$

$$f(z) = |z|$$

computable ?

$$f^* = \alpha \circ f \circ \alpha^{-1} : \mathbb{N} \to \mathbb{N}$$

$$f^*(m) = \alpha \circ f \circ \alpha^{-1}(m) = \begin{cases} m \text{ even} & \alpha\left(f\left(\frac{m}{2}\right)\right) = \alpha\left(\frac{m}{2}\right) = m \\ m \text{ odd} & \alpha\left(f\left(-\frac{m+1}{2}\right)\right) = \alpha\left(\frac{m+1}{2}\right) = m+1 \end{cases}$$

$$= \begin{cases} m & \text{if } m \text{ is even} \\ m+1 & \text{if } m \text{ is odd} \end{cases}$$

URM- computable