Differentiation of measures

(Theorem)

Let μ be a Radon measure, $\mu \ll \mathcal{L}$ (Lebesgue)

then

$\forall$ for $\underline{\text{a.e}}$ $x \in \mathbb{R}^n$ $\exists$ finite

$$\lim_{r \to 0^+} \frac{\mu(B(x,r))}{|B(x,r)|} = f(x)$$

+ f is measurable (Lebesgue)

WE CALL f the DENSITY of the MEASURE

μ Radon
ν Radon
$\mu \ll \nu$
$$\lim_{r \to 0^+} \frac{\mu(B(x,r))}{\nu(B(x,r))} = f(x)$$

Corollary ((Vitali) – Lebesgue theorem)

Let $f \in L^1_{loc}(\mathbb{R}^n)$ $\quad (f \in L^1(K) \quad K \subset\subset \mathbb{R}^n)$

- for a.e $x \in \mathbb{R}^n$

$$f(x) = \lim_{r \to 0^+} \frac{1}{|B(0,r)|} \int_{B(x,r)} f(y)\,dy$$

Corollary of the prev. thm because if $f \in L^1_{loc}(\mathbb{R}^n)$, $f \geq 0$

$A \in \mathcal{B}$ $\quad \mu(A) := \int_A f(y)\,dy$ $\qquad \mu << \mathcal{L}$

μ Radon.

ν is Radon $f \in L^{1,\nu}_{loc}(\mathbb{R}^n)$ for a.e x (in ν-sense)

$$f(x) = \lim_{r \to 0^+} \frac{1}{\nu(B(x,r))} \int_{B(x,r)} f(y)\, d\nu(y)$$

($f \geq 0$ $f \in L^{1,\nu}_{loc}$ ν is Radon

$$\mu(A) := \int_A f(y)\, d\nu(y)$$

Corollary (of the corollary)

$f \in L^p_{loc}(\mathbb{R}^n)$ $\quad p \in [1, +\infty)$

$$\lim_{r \to 0^+} \frac{1}{|B(0,r)|} \int_{B(x,r)} |f(y) - f(x)|^p \, dy = 0$$

for a.e $x \in \mathbb{R}^n$

The proofs of these theorems are based on <u>covering theorems</u>

Covering theorem (Vitali ~1908, Besicovitch ~1948)

Let μ be a Radon measure

$\mathcal{F}$ collection of closed balls (non degenerate $\Rightarrow$ radius >0)

$A = \{\text{centers of balls in } \mathcal{F}\}$ $\mu(A) < +\infty$.

Assume $\forall a \in A$ $\inf\{r > 0 \;\; \overline{B(a,r)} \in \mathcal{F}\} = 0$

then $\exists\, \mathcal{G} \subseteq \mathcal{F}$ COUNTABLE and containing DISJOINT BALLS

such that $\mu\left(A \cap U \setminus \bigcup_{B \in \mathcal{G}} \overline{B}\right) = 0$

(how to use?: fix U open sets $\mu(U) < +\infty$

fix $\delta > 0$ $\quad \mathcal{F} = \{ \overline{B}$ such that $\overline{B} = \overline{B}(a, r)$; for some $a \in U$, $r \leq \delta$, and $\overline{B(a,r)} \subseteq U \}$.

So $\exists\, \mathcal{G} \subseteq \mathcal{F}$ (exists $a_n \in U$, $r_n \leq \delta$

$$\overline{B(a_n, r_n)} \cap \overline{B(a_m, r_m)} = \emptyset$$

such that $\mu\left(U \setminus \bigcup_{n=1}^{\infty} \overline{B(a_n, r_n)} \right) = 0$)

Sketch of proof of differentiation theorem.

fix $K \subset\subset \mathbb{R}^n$ $\alpha > 0$

1 $A \subseteq \{ x \in K \quad \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{|B(x,r)|} \geq \alpha \}$

then $\mu(A) \geq \alpha |A|$

2 $A \subseteq \{ x \in K \quad \liminf_{r \to 0^+} \frac{\mu(B(x,r))}{|B(x,r)|} \leq \alpha \}$

then $\mu(A) \leq \alpha |A|$

proof of 2, proof of (1) will be similar.

proof of (2)

$$\mathcal{F}_\varepsilon = \{ \overline{B} \quad \overline{B} = \overline{B}(a,r) \quad a \in A, \quad \overline{B}(a,r) \subseteq U, \quad \mu(\overline{B}(a,r)) \le (\alpha+\varepsilon)\,|B(a,r)| \}$$

So $\inf \{ r \;\; \overline{B}(a,r) \in \mathcal{F}_\varepsilon \} = 0$, indeed $a \in A \Rightarrow$

$$\Rightarrow \liminf_{r \to 0^+} \frac{\mu(\overline{B}(a,r))}{|B(a,r)|} \le \alpha < \alpha + \varepsilon \Rightarrow \exists\, r_n \to 0^+ \quad \frac{\mu(\overline{B}(a,r_n))}{|B(a,r_n)|} \le \alpha + \varepsilon$$

We may apply the covering theorem $\Longrightarrow \exists \; B_n$ disjoints

$$\mu\Big(A \setminus \bigcup_n \overline{B}_n\Big) = 0 \qquad \overline{B}_n \subseteq U,$$

$$\mu(A) \le \sum_n \mu(\overline{B}_n) \le (\alpha+\varepsilon) \sum_n |\overline{B}_n| \le$$

$\downarrow$ σ-additivity

$$\le (\alpha+\varepsilon)\,|U|$$

$\forall\, U$ open with $U \supseteq A$. $\mu(A) \le (\alpha+\varepsilon)|U|$

since Lebesgue measure is REGULAR $\Rightarrow \mu(A) \le (\alpha+\varepsilon)\,|A|$

$\Rightarrow$ tending $\varepsilon \to 0$ $\quad \mu(A) \leq \alpha |A|$

$(\forall K \subset\subset \mathbb{R}^n$ compact $\Longrightarrow$ $A = \{ x \in \mathbb{R}^n \;\; \liminf_{r \to 0^+} \frac{\mu(B(x,r))}{|B(x,r)|} \leq \alpha \}$

$\Rightarrow \mu(A) \leq \alpha |A|$

By (1) and (2) we deduce

$\left\{ x \mid \liminf_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} < \limsup_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} \right\}$ has measure 0.

Let $a < b$ $\quad D = \{ x \mid \underbrace{\liminf_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} \leq a}_{\text{prop } 2} < b \leq \underbrace{\limsup_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|}}_{\text{prop}(1)}$

prop 2 $\quad \mu(D) \leq a |D|$ $\qquad$ prop(1) $\quad \mu(D) \geq b |D|$

$b |D| \leq \mu(D) \leq a |D|$ $\Longrightarrow$ since $a < b \Longrightarrow |D| = 0$

$C = \{ x \ \limsup_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|} = +\infty \}$ has measure 0

$\downarrow$

$$C \subseteq \left\{ x \ \limsup \frac{\mu(B(x,r))}{|B(x,r)|} \geq \alpha \right\} \quad \text{for } \alpha > 0$$

$\Rightarrow$ prop 2 $\Rightarrow$ $\mu(C) \geq \alpha |C| \Rightarrow |C| \leq \frac{1}{\alpha}\mu(C)$

$\Rightarrow$ sending $\alpha \to +\infty \Rightarrow |C| = 0$.

So we get for a.e. x, $f(x) := \lim_{r \to 0^+} \frac{\mu(B(x,r))}{|B(x,r)|}$

is finite and well defined.

For the proof of measurability of f, see EVANS GARIEPY ch. 1.6.

Ex (on the INTEGRATION by SPHERE formula)

Let $B(0,1) \subsetneq \mathbb{R}^n$, we define $\omega_n := |B(0,1)|$

ω_n is the volume of n-dimensional ball of radius 1.

So for $x \in \mathbb{R}^n$, $r > 0$ $\quad |B(x,r)| = \omega_n r^n$ (by invariance by translation and n-homogeneity)

$$\omega_n = \int_{B(0,1)} dx = \int_{\mathbb{R}^n} \underbrace{\chi_{B(0,1)}}(x)\, dx =$$

$$\chi_{B(0,1)}(x) = \begin{cases} 1 & x \in B(0,1) \\ 0 & x \notin B(0,1) \end{cases}$$

$=$ integration on spheres formula $= \displaystyle\int_0^1 r^{n-1} \int_{S^{n-1}} dS\, dr$

$S^{n-1} = \{x \in \mathbb{R}^n \quad |x| = 1\}$

$$= \int_0^1 r^{n-1} dr \; \mathcal{A}(S^{n-1})$$

$\mathcal{A}(S^{n-1}) = \text{area} = n\omega_n$

$$= \frac{1}{n} \mathcal{A}(S^{n-1})$$

consider $f(x) = e^{-|x|^2} \quad x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \text{by Fubini Tonelli} = \prod_{i=1}^{n} \left(\int_{-\infty}^{+\infty} e^{-|x_i|^2} dx_i \right) =$$

$$= \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \right)^n$$

$$\int_{\mathbb{R}^2} e^{-|x|^2} dx = \text{polar coordinates} = \int_0^{+\infty} e^{-\rho^2} \rho \, d\rho \cdot \int_0^{2\pi} d\theta =$$

$x = (x_1, x_2)$ $\quad x_1 = \rho \cos\theta$
$\quad x_2 = \rho \sin\theta$

$$= 2\pi \int_0^{+\infty} \rho \, e^{-\rho^2} d\rho = \pi. \quad \Rightarrow \left(\int_{-\infty}^{+\infty} e^{-t^2} dt \right)^2 = \int_{\mathbb{R}^2} e^{-|x|^2} dx = \pi$$

$$\Rightarrow \int_{\mathbb{R}^n} e^{-|x|^2} dx = \pi^{n/2}$$

$$\pi^{n/2} = \int_{\mathbb{R}^n} e^{-|x|^2} dx = \text{integration by shere form.} = \int_0^{+\infty} r^{n-1} e^{-r^2} \mathcal{H}(S^{n-1}) dr$$

$$= n\omega_n \int_0^{+\infty} r^{n-1} e^{-r^2} dr = n\omega_n \int_0^{+\infty} s^{\frac{n-1}{2}} e^{-s} \frac{s^{-1/2}}{2} ds =$$

$s = r^2$

$$= \frac{n\omega_n}{2} \int_0^{+\infty} s^{\frac{n}{2}-1} e^{-s} ds = \frac{n\omega_n}{2} \Gamma\left(\frac{n}{2}\right) = \omega_n \Gamma\left(\frac{n}{2}+1\right)$$

$$\omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

where $\Gamma(x) =$ GAMMA FUNCTION by Euler $= \int_0^{+\infty} e^{-s} s^{x-1} ds$

$x\Gamma(x) = \Gamma(x+1)$

$\Gamma(n) = (n-1)!$