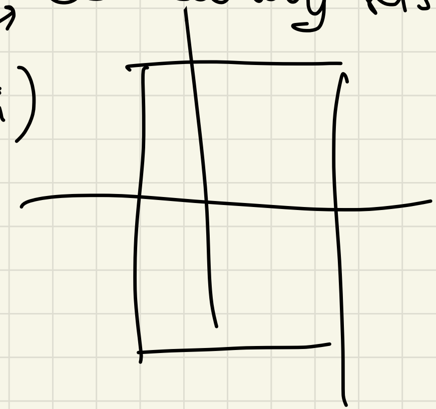
How to construct a measure?

- one starts to define a "measure" in a family of elementary sets

e.g. rectangular ~~sp~~ $\prod_{i=1}^{n} (a_i, b_i)$

elementary sets



$$\tilde{\mu}\left(\prod_{1}^{n} (a_i; b_i)\right) = \prod_i |b_i - a_i|$$

$A \in \mathcal{P}(\mathbb{R}^n)$ $\quad (A \subseteq \mathbb{R}^n)$

$$\mu^*(A) := \inf \left\{ \sum_{i=1}^{\infty} \tilde{\mu}(E_i) \quad A \subseteq \bigcup_{i=1}^{\infty} E_i \right\}$$

$\mu^{\sim}$

(E_i elementary set (RECTANGULAR SET))

μ^* is called OUTER MEASURE

$\mu^*(\emptyset) = 0$

- NOT σ-additive

- A_i disjoint sets

$$\mu^*(\cup_i A_i) \leq \sum_i \mu^*(A_i)$$

SUB-ADDITIVE

Definition : $A \in \mathcal{P}(\mathbb{R}^n)$ $(A \subseteq \mathbb{R}^n)$ is MEASURABLE w.r.t μ^* if

$\forall B \subseteq \mathbb{R}^n$ $(\forall B \in \mathcal{P}(\mathbb{R}^n))$

$$\mu^*(B) = \mu^*(B \setminus A) + \mu^*(A \cap B)$$

<u>CARATHEODORY THEOREM</u> μ^* OUTER MEASURE

1) family of MEASURABLE sets w.r. to μ^* is a Σ-algebra, $\mathcal{M}$, $\mu^*|_{\mathcal{M}}$ is a measure (σ-additive)

2) if $\forall A, B \subseteq \mathbb{R}^n$ with $d(A,B) > 0$

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

Then the σ-algebra of μ^* measurable sets contains the Borel σ-algebra

(Borel sets are measurable with respect to μ^*)

Extension

Signed Borel measure $\quad \mu : \mathcal{B} \longrightarrow [-\infty, +\infty]$

$\mathcal{B}$ = Borel σ-algebra

$\mu(\phi)=0 \qquad \mu\left(\bigcup_i^{\infty} A_i\right) = \sum_i^{\infty} \mu(A_i) \qquad A_i \cap A_j = \phi$ $\quad i \neq j$

μ cannot assume both the value $+\infty$ and $-\infty$.

($A \in \mathcal{B}$ is a NULL SET if $\mu(B)=0 \quad \forall B \subseteq A$, $B \in \mathcal{B}$)

$A \in \mathcal{B}$ is a POSITIVE SET if $\mu(B) \geq 0 \quad \forall B \subseteq A$)

...

vector valued measures:

$$\mu : \mathcal{B} \longrightarrow \overline{\mathbb{R}}^d \qquad [-\infty, +\infty]^d$$

$$A \longmapsto (\mu_1(A), \dots, \mu_d(A))$$

every μ_i is a signed measure.

Let μ be a (signed/vector valued) measure

the **total variation measure** is the

SMALLEST POSITIVE MEASURE ($\nu : \mathcal{B} \to [0, +\infty]$)

such that $\forall B \in \mathcal{B}$ $\quad |\mu(B)| \leq \nu(B)$

(If μ is POSITIVE MEASURE then it coincides with the TOTAL VARIATION m.

TOTAL VARIATION MEASURE associated to μ is denoted $|\mu|$

$\downarrow$

operatively can be computed as follows

$\forall B \in \mathcal{B}$

$$|\mu|(B) := \sup \left\{ \sum_{i}^{\infty} |\mu(B_i)| \quad \begin{array}{l} B_i \in \mathcal{B} \\ B = \cup_i B_i \\ B_i \cap B_j = \phi \end{array} \right\}$$

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signed measure (taking values in $[-\infty, +\infty]$)

JORDAN DECOMPOSITION $\quad \mu^+, \mu^-$ POSITIVE MEASURES

$$\mu = \mu^+ - \mu^- \qquad \mu^+ = \frac{\mu + |\mu|}{2} \qquad \mu^- = \frac{|\mu| - \mu}{2}$$

<u>Def</u> Support of a measure

μ $\quad \mathrm{supp}(\mu) = \{ \overline{x \in \mathbb{R}^n \text{ such that}}$
$\overline{|\mu|(B(x,r)) > 0 \quad \forall r > 0} \} = C$

$\downarrow$

$|\mu|(\mathbb{R}^n \setminus C) = 0$

<u>Definition</u> μ signed / vector valued measure

λ POSITIVE MEASURE $\quad (\lambda = \mathcal{L}$ Lebesgue$)$

$\mu \ll \lambda$ (μ is ABSOLUTELY CONTINUOUS with respect to λ)

If $\forall B \in \mathcal{B}$ s. that $\lambda(B) = 0$ it holds $|\mu|(B) = 0$

$\mu \perp \lambda$ (μ singular with respect to λ)

iff $\exists A, B \in \mathcal{B}$ $A \cap B = \emptyset$ $A \cup B = \mathbb{R}^n$

s.that $\underbrace{|\mu|(A) = 0}_{}$ $\underline{\lambda(B) = 0}$

equivalent to say $\cancel{\mu}$ A is a null set of μ.

$\delta_0 \perp \mathcal{L}$ $(\mu^+ \perp \mu^-)$

$$\delta_0(A) = \begin{cases} 1 & 0 \in A \\ 0 & 0 \notin A \end{cases}$$

$\mathcal{L}$ Lebesgue

DIRAC

$$\mathbb{R}^n = (\mathbb{R}^n \setminus \{0\}) \cup \{0\}$$

Lebesgue - Radon - Nikodym Theorem

Let μ be a σ-finite vector valued measure (takes values in $\mathbb{R}^d$, $d \geq 1$)

(or simply a σ-finite signed measure)

Let λ be a POSITIVE RADON MEASURE (take $\lambda = \mathcal{L}$ LEBESGUE)

$\exists!$ ρ, ν σ-finite vector measures such

that $\rho \ll \lambda$ $\quad \nu \perp \lambda$ $\quad \mu = \rho + \nu$

$\exists f : \mathbb{R}^n \to \mathbb{R}^d$ DENSITY.

$\forall E \in \mathcal{B}$ $\quad \rho_i(E) = \int_E f_i(x)\, d\lambda(x)$ $\quad i = 1 \dots d$

Obs. μ vector valued (Taking values in $\mathbb{R}^d$)

$\|\mu\|$ total variation measure

(assume that they are locally finite
= they are finite on compact sets on $\mathbb{R}^n$)

($|\mu|$ is RADON)

$\mu << |\mu|$ by the theorem $\exists f : \mathbb{R}^n \to \mathbb{R}^d$

"$\mu = f \cdot |\mu|$" $\quad \mu_i(E) = \int_E f_i(x) \, d|\mu|$ $\quad \forall E \in \mathcal{B}$

$|f| = 1$ a.e. x.

Observation: Let $E \in \mathcal{B}$

and define $\nu_E(A) := \underbrace{|E \cap A|}_{\text{Lebesgue measure}} \quad \forall A \in \mathcal{B}$

ν_E is a positive measure (Radon) and

$\nu_E << \mathcal{L}$ = Lebesgue measure

it has a density $f_E : \mathbb{R}^n \to [0, +\infty]$

$$f_E(x) := \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|} = \lim_{r \to 0^+} \frac{|E \cap B(x,r)|}{|B(0,1)| \, r^n}$$

this limit exists for a.e x.

$\{x \mid f_E(x) = 1\} \subseteq E$ is the MEASURE THEORETIC interior of E

$\downarrow$

(actually $f_E(x) = 1$ for a.e $x \in E$)

and $f_E(x) = 0$ for a.e $x \in \mathbb{R}^n \setminus E$.

MEASURE THEORETIC BOUNDARY of E is

$$\{ x \in \mathbb{R}^n, \quad f_E(x) \in (0,1) \}.$$