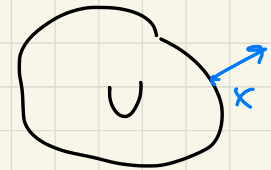
Lesson WEDNESDAY OCTOBER 9

is cancelled.



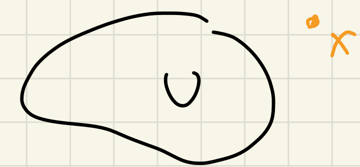
$U \subseteq \mathbb{R}^n$ open bdd set is of class C^k $k \geq 1$

if locally its boundary ∂U is parametrized as a graph of a function in $C^k(\mathbb{R}^{n-1})$

and U coincides locally with the epigraph.

at every point $x \in \partial U$ we may define a vector $\nu(x)$ $|\nu(x)|=1$ ν is the exterior normal to x at U.

SIGNED DISTANCE from $\bar{U}$



$\forall x \in \mathbb{R}^n \qquad d_S : \mathbb{R}^n \to \mathbb{R}$

$$d^S_U(x) = \text{dist}(x, \bar{U}) - \text{dist}(x, \mathbb{R}^n \setminus \bar{U})$$

$x \in \partial U \Rightarrow d^S(x) = 0$

$x \in \bar{U} \qquad d^S(x) < 0 \qquad\qquad x \in \mathbb{R}^n \setminus \bar{U} \qquad d^S(x) > 0$

this function is 1-Lipschitz continuous (it is Lipschitz continuous, with Lipschitz constant $= 1$)

$x \in \partial U \qquad \nu(x) = \nabla d_S(x)$

Theorem (Elliptic PDEs, Gilbarg Trudinger)

Let U a bdd open set of class C^k with $k \geq 2$.

$\exists\, r > 0$ $(\partial U)_r := \{ x \in \mathbb{R}^n \; -r < d^S_U(x) < r \}$ OPEN SET

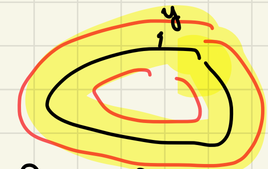
$\exists\, r > 0$ (depend on U)

such that $d^S_U \in C^k((\partial U)_r)$

$\nabla d^S_U(x) \in C^{k-1}((\partial U)_r ; \mathbb{R}^n)$

$\Rightarrow$ $\nu(x)$ CAN BE EXTENDED in $(\partial U)_r$ to a VECTOR FIELD OF CLASS C^{k-1}

idea

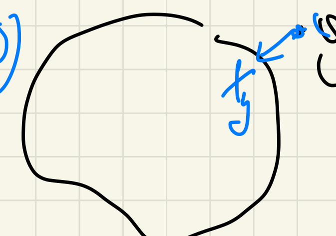


if I have a bdd op. set U of class $\underline{\mathcal{C}^2}$ or better ($\mathcal{C}^k$, $k>2$) there $\exists\, r>0$ such that $\underbrace{\forall y \in (\partial U)_r}$ $= \{ z \in \mathbb{R}^n : -r < d^S_U(z) < r \}$

$\downarrow$

$\exists!\, x \in \partial U \quad |x-y| = d(y, \partial U) \left(= \min_{z \in \partial U} |y-z| \right)$

$\forall y \in (\partial U)_r \to \exists!\, x_y \in \partial U \quad |x_y - y| = d(y, \partial U)$



$$\boxed{y = \underline{x_y} + d_S(y) \cdot \nu(x_y)}$$

in order to have a unique projection on the boundary (at least for $x \in (\partial U)_r$ for r sufficiently small)

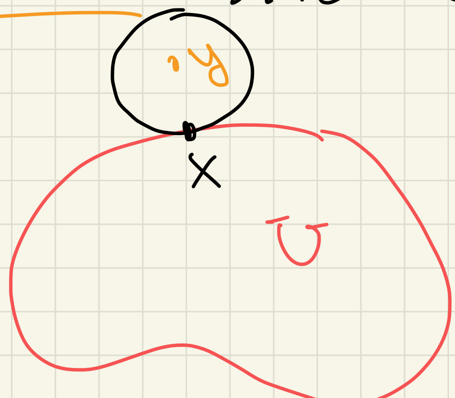
$\downarrow$

we need the following geom. conditions on U

- EXTERIOR BALL CONDITION $\to$ $\forall x \in \partial U$ $\exists y \in \mathbb{R}^n \setminus U$ $\exists \delta > 0$

$$B(y,\delta) \subseteq \mathbb{R}^n \setminus U$$

$$\overline{B(y,\delta)} \cap \overline{U} = \{x\}$$



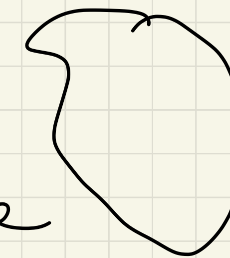
Convex sets have always the exterior ball cond. $\forall \delta > 0$

• INTERIOR BALL CONDITION

$\forall x \in \partial U \quad \exists y \in U$ and $\delta > 0$

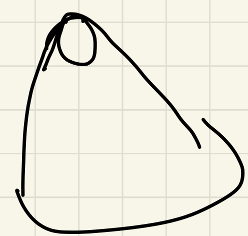
such that $\quad B(y, \delta) \subseteq U$

$$\overline{B(y, \delta)} \cap \mathbb{R}^n \setminus U = \{x\}$$



δ is taken UNIFORM on ∂U

δ is related to a bound on the curvature of ∂U.



$$U = \{(x_1, x_2) \quad x_1 \in \mathbb{R} \quad x_2 \leq x_1^2 \lg|x_1| \}$$

bdd sets which satisfies $\overset{\delta \text{ UNIFORM}}{(\text{Uniform})}$ exterior and interior ball conditions are set of class $C^{1,1}$ (locally parametrized with functions of class $C^{1,1}$

$$\left(\gamma \in C^1, \ \nabla \gamma \in \underset{\text{LIPSCHITZ}}{C^{0,1}} \right)$$

(Abstract) Measure theory

Geometric measure theory (BESIKOVITCH '30, DE GIORGI ENNIO, '50, FEDERER, ALMGREN ...)

- Radon measures
- Hausdorff measures

$X = \mathbb{R}^n$

X = locally compact ($\forall x$ has a compact neighborhood)

topological Hausdorff space (separated)

which is σ-compact (countable union of compact)

σ-algebra on X is a subset of $\mathcal{P}(X)$ containing ϕ, closed by countable union and passage to the complement

(BIGGEST is $\mathcal{P}(X)$
SMALLEST is $\{\phi, X\}$)

$\mathcal{B}$ = Borelian σ-algebra (smallest σ-alg. containing all the open sets of X)

a Borel measure is a function

$$\mu : \mathcal{B} \longrightarrow [0, +\infty]$$

$$\mu(\emptyset) = 0 \qquad \mu\left(\bigcup_{1}^{+\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

$A_i \in \mathcal{B}$ $\quad A_i \cap A_j = \emptyset \quad i \neq j$

$\boxed{\sigma\text{-additive}}$

μ is finite if $\mu(X) < +\infty$

μ is σ-finite if $X = \bigcup_{i=1}^{\infty} E_i$ with $\mu(E_i) < +\infty$

Def A Borel measure μ is called a RADON MEASURE if $\mu(K) < +\infty \quad \forall K$ COMPACT in $\mathbb{R}^n$ (LOCALLY FINITE) obs if μ is Borel and finite ($\mu(X) < +\infty$) $\Rightarrow$ μ is RADON

if μ is Radon either μ is finite or μ is σ-finite

$$\mathbb{R}^n = \bigcup_{N=1}^{+\infty} \overline{B(0,N)}$$

μ Radon $\Rightarrow$ $\mu(\overline{B(0,N)}) < +\infty$

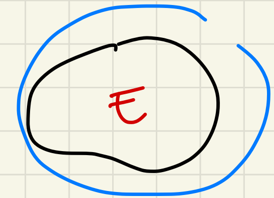
The Lebesgue measure $\mathcal{L}$ is the UNIQUE Radon measure on $\mathbb{R}^n$ which is TRANSLATION INVARIANT and HOMOGENEOUS of degree n.

$\mathcal{L}(A) = \mathcal{L}(A+x) \quad \forall x \in \mathbb{R}^n$

$\mathcal{L}(\lambda A) = \lambda^n \mathcal{L}(A)$

<u>Proposition</u> If μ is a Radon measure then it is REGULAR

$\forall E \in \mathcal{B}$ (Borel set)



$$\mu(E) = \inf \{ \mu(U) \, , \; U \text{ open sets } U \supseteq E \}$$

$$= \sup \{ \mu(K) \quad K \text{ compact } K \subseteq E \}$$

μ Radon I can define INTEGRATION with respect to μ.

ϕ simple functions $\quad \underbrace{\phi = \sum_{1}^{N} a_i \chi_{E_i}}$

$a_i \geq 0 \qquad E_i \in \mathcal{B}$

$f \geq 0$

$$\int_{\mathbb{R}^n} f(x)\, d\mu = \sup \left\{ \sum_i a_i \mu(E_i), \quad 0 \leq \phi \leq f \quad \phi \text{ simple} \right\}$$

$$\phi = \sum a_i \chi_{E_i}$$