COMPUTABILITY (07/10/2024)

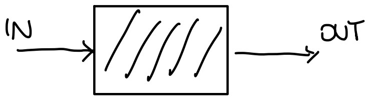
effective procedure (algorithm)
$\downarrow$
computable function

$\Big\} \to$ existence of non computable functions

* Effective procedure

sequence of elementary steps

input data $\rightsquigarrow$ output data



deterministic
$\downarrow$
function (partial) $\quad f: \{\text{inputs}\} \to \{\text{outputs}\}$

Def. A function is computable when there exists an algorithm which induces the function

* $GCD(x,y)$ = greatest common divisor (Euclid's)
* $f(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$

- $h(n) = n^{th}$ prime number
- $g(n) = n^{th}$ digit of π
- $f(n) = \begin{cases} 1 & \text{if in } \pi \text{ there are exactly } n \text{ digits } 5 \\ 0 & \text{otherwise} \end{cases}$

if $\pi = 3,14 \ldots\ldots \; 7\,5\,5\,5\,5\,4 \ldots\ldots$
$\downarrow$
$f(4) = 1$

idea : - compute all digits of π

- check if there are n digits 5 in a row

NOT AN ALGORITHM!

Is f computable ? I don't know!

* $g(n) = \begin{cases} 1 & \text{if in } \pi \text{ there is a sequence of 5's of length at least } n \\ 0 & \text{otherwise} \end{cases}$

if $\pi = 3,14 \ldots\ldots 7\ 5\ 5\ 5\ 5\ 4 \ldots\ldots$

$\downarrow$

$g(4) = 1$

$g(3) = 1$

$g(2) = 1$

$g(1) = 1$

$g(0) = 1$

$\forall n \in \mathbb{N}$ if $g(n) = 1$

$\Rightarrow \forall m \le n \quad g(m) = 1$

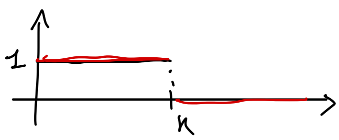
take $k = \sup\{ n \mid$ there is in π a sequence of n consecutive 5's $\}$

$\rightsquigarrow$

two possibilities

* k finite

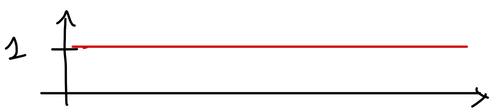
$g(n) = \begin{cases} 1 & n \le k \\ 0 & \text{otherwise} \end{cases}$



function $g(n)$:
  if $n \le k$:
    return 1
  else: return 0

* k infinite

$g(n) = 1 \quad \forall n$



function $g(n)$:
  return 1

Can we use the same trick for f?

- $f(n) = \begin{cases} 1 & \text{if in } \pi \text{ there are exactly } n \text{ digits } 5 \\ 0 & \text{otherwise} \end{cases}$

we define $A = \{ n \mid \text{there are exactly } n \text{ digits } 5 \text{ in } \pi \}$

and

```
function f(n):
  if n ∈ A:
      return 1
  else
      return 0
```

potentially infinite program!

* Existence of non computable functions

* Properties of a "reasonable" algorithm

- → finite length
- → there exists a computing agent which executes the algorithm
  - → memory (unbounded)
  - → discrete steps, deterministic, not probabilistic
  - → finite limit to the number and power instructions
  - → the computation can
    - terminate after a finite amount of steps → OUTPUT
    - diverge (never terminate) → NO OUTPUT

* Math notation

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

A, B $\quad\quad A \times B = \{(a,b) \mid a \in A,\ b \in B\}$

$$A^m = \underbrace{A \times \cdots \times A}_{m \text{ times}}$$

* relations

$r \subseteq A \times B$

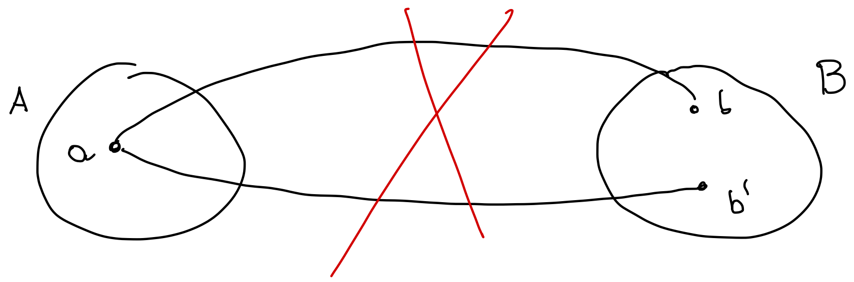
m divides n $\quad\quad$ $\text{divisor} = \{(m, m \times k) : m, k \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$

* function (partial)

$f : A \to B$

special relation

$\forall a \in A \quad \forall b, b' \in B \quad (a,b), (a,b') \in f$ then $b = b'$



$\text{dom}(f) = \{a \in A \mid \exists b \in B \ (a,b) \in f\}$

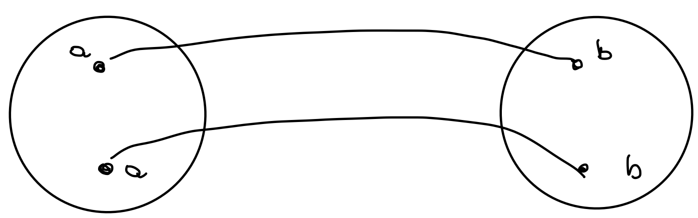
instead of $(a,b) \in f$ we write $f(a) = b$

if $a \in \text{dom}(f)$ $\quad\quad f(a)\downarrow$

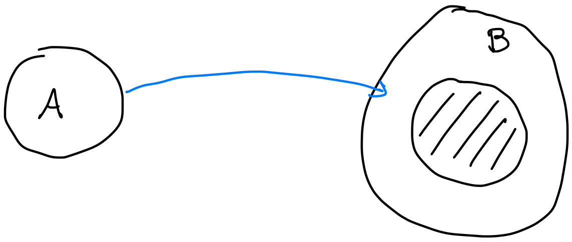
if $a \notin \text{dom}(f)$ $\quad\quad f(a)\uparrow$

* Cardinality

A, B $\quad\quad |A| = |B|$ if there is $f : A \to B$ bijective

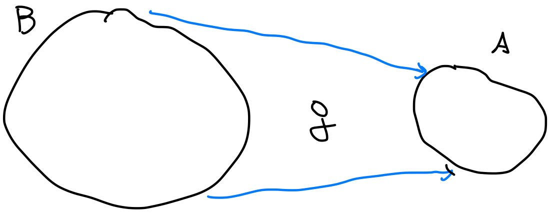


$|A| \leq |B|$ if there is $f: A \to B$ injective



"equivalent"

if there is a surjective function $g: B \to A$



- $\underline{A \text{ countable (denumerable)}}$

  if $|A| \leq |\mathbb{N}|$ i.e. there is $f: \mathbb{N} \to A$ surjective

$$\underbrace{f(0) \quad f(1) \quad f(2) \quad f(3) \quad \cdots\cdots}_{A}$$

- A, B countable then $A \times B$ countable

  <u>idea</u>:

$$\begin{array}{lllll} A & a_0 & a_1 & a_2 & \cdots \\ B & b_0 & b_1 & b_2 & \cdots \end{array}$$

$$\begin{array}{c|cccc} & a_0 & a_1 & a_2 & \\ \hline b_0 & (a_0,b_0) & (a_1,b_0) & (a_2,b_0) & \cdots \\ b_1 & (a_0,b_1) & (a_1,b_1) & (a_2,b_1) & \cdots \\ b_2 & (a_0,b_2) & (a_1,b_2) & (a_2,b_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

dove taie

- $A_0, A_1, A_2 \ldots$ countable collection of countable sets $(|A_i| \leq |\mathbb{N}|)$

  then $\bigcup_{i \in \mathbb{N}} A_i = A_0 \cup A_1 \cup \cdots$ is countable

* Existence of non computable functions

we restrict to unary functions on $\mathbb{N}$ (partial)

$$\mathcal{F} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$$

Fix a model of computation $\to$ set of algorithms $\mathcal{A}$

given $A \in \mathcal{A}$ this computes $f_A : \mathbb{N} \to \mathbb{N}$

Functions computable in $\mathcal{A}$

$$\mathcal{F}_{\mathcal{A}} = \{ f \in \mathcal{F} \mid \text{there is } A \in \mathcal{A} \text{ s.t. } f = f_A \}$$
$$= \{ f_A \mid A \in \mathcal{A} \}$$

Clearly

$$\mathcal{F}_A \subsetneq \mathcal{F}$$

?

yes

* An algorithm is a sequence of instructions from I (finite set of instructions of the model)

$$\mathcal{A} = I \cup I \times I \cup I \times I \times I$$
$$= \bigcup_{i \geq 1} I^i$$

$\nwarrow$ countable union of countable sets $\Rightarrow$ countable

$$|\mathcal{A}| \leq |\mathbb{N}|$$

Now

$$\mathcal{A} \longrightarrow \mathcal{F}_A$$
$$A \longmapsto f_A$$

surjective

$$\boxed{|\mathcal{F}_A| \leq |\mathcal{A}| \leq |\mathbb{N}|}$$

* <u>The set $\mathcal{F}$ of all functions is not countable</u>

Why? Assume $\mathcal{F}$ is countable and enumerate $\mathcal{F}$

	f_0	f_1	f_2	f_3 - - - - -
0	$f_0(0)$	$f_1(0)$	$f_2(0)$	
1	$f_0(1)$	$f_1(1)$	$f_2(1)$	
2	$f_0(2)$	$f_1(2)$	$f_2(2)$	

systematically change the diagonal function

defined $\rightsquigarrow$ $\uparrow$

$\uparrow$ $\rightsquigarrow$ defined (e.g. 0)

define $d : \mathbb{N} \to \mathbb{N}$

$$d(n) = \begin{cases} \uparrow & \text{if } f_n(n)\downarrow \\ 0 & \text{if } f_n(n)\uparrow \end{cases}$$

now $d \in \mathcal{F}$

$d \neq f_n \quad \forall n \quad$ since $\quad d(n) \neq f_n(n)$

$\hookrightarrow$ if $f_n(n)\uparrow \Rightarrow d(n) = 0$

if $f_n(n)\downarrow \Rightarrow d(n)\uparrow$

contradiction

$\Downarrow$

$$\boxed{|\mathcal{F}| > |\mathbb{N}|}$$

Putting things together

$$\left.\begin{array}{l} \mathcal{F}_A \subseteq \mathcal{F} \\ |\mathcal{F}_A| \le |\mathbb{N}| < |\mathcal{F}| \end{array}\right\} \Rightarrow \mathcal{F}_A \subsetneq \mathcal{F}$$

How many non-computable functions

$|\mathcal{F} \setminus \mathcal{F}_A| > |\mathbb{N}|$