PARTITIONS of UNITS subordinated to an open cover.

$A \subseteq \bigcup_{i=1}^{\infty} U_i$ open cover of A

set

U_i are open set

$(\xi_i)_{i\in\mathbb{N}}$ is a PARTITION of UNITS subordinated to the open cover $\{U_i\}_{i\in\mathbb{N}}$ if

1) $\xi_i \in C_c^\infty(U_i) \subseteq C_c^\infty(\mathbb{R}^n)$ $\forall i$

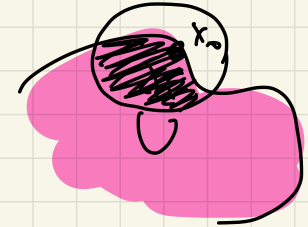
2) $0 \leq \xi_i(x) \leq 1$ $\forall x \in \mathbb{R}^n$

3) $\sum_{1}^{\infty} \xi_i(x) = 1$ $\forall x \in A$ (locally finite sum)

$\forall x \in A$ $\exists N_x \in \mathbb{N}$

$\xi_i(x) \neq 0$ only for $i \in I$ $\#I = N_x$

Preliminaries from Calculus.



$U \subseteq \mathbb{R}^n$ open set $\quad n > 1$

We say that is a ~~set~~ open set of class C^k $\left(\begin{matrix} k \in \mathbb{N} \\ k \geq 1 \end{matrix} \right)$

if $\forall x_0 \in \partial U$ $\exists r > 0$ and

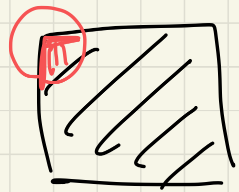
TOPOLOGICAL BOUNDARY of U

$\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ $\quad \gamma \in C^k(\mathbb{R}^{n-1})$ such that

(UP TO A ROTATION and UP TO REORDERING VARIABLES.)

$$\boxed{U \cap B(x_0, r) = \{ (x_1 \ldots x_n) \in \mathbb{R}^n, \; x_n < \gamma(x_1 \ldots x_{n-1}) \}}$$

U is a Lipschitz set if the previous def. holds for $\gamma \in C^{0,1}(\mathbb{R}^{n-1})$
(LIPSCHITZ)

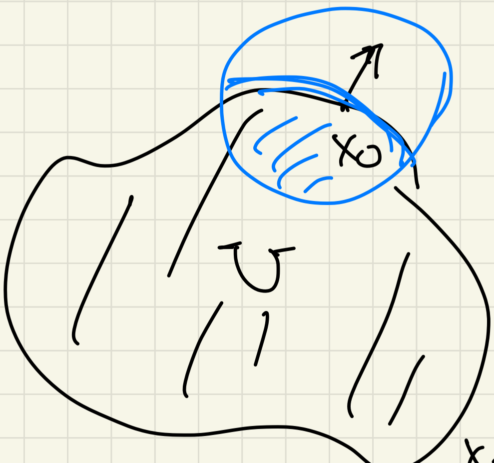
ex  square in $\mathbb{R}^2$.

U is C^∞, or analytic C^ω, if $\gamma \in C^\infty$ or $\gamma \in C^\omega$.

If $U \subseteq \mathbb{R}^n$ is a (bdd) open set of class C^1 then at every $x_0 \in \partial U$ I can define the

EXTERIOR NORMAL (vector)

$\underline{\nu(x_0)} \in \mathbb{R}^n$ $\qquad |\nu(x_0)| = 1$ $\qquad x_0 = ((x_0)_1, \dots (x_0)_n) \in \mathbb{R}^n$



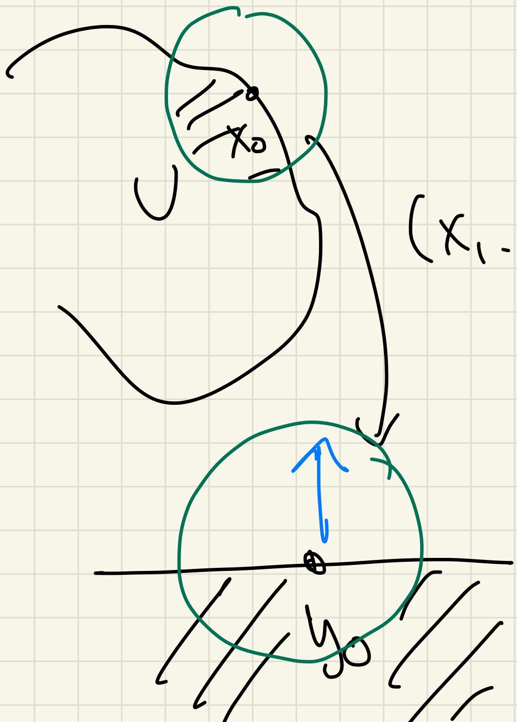
$r > 0$, $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ $\quad \gamma \in C^1$

$U \cap B(x_0, r) = \{ \underbrace{(x_1, \dots x_n) \quad x_n < \gamma(x_1 \dots x_{n-1})} \}$

$$\nu(x_0) = \frac{\left(-\nabla\gamma((x_0)_1, (x_0)_2, \dots, (x_0)_{n-1}), 1 \right)}{\sqrt{1 + |\nabla\gamma(\cdot\cdot)|^2}}$$

$(x_0)_n = \gamma((x_0)_1, \dots, (x_0)_{n-1})$

a typical procedure will be to locally flatten the boundary around x_0



$$\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad C^1 \text{ diffeo.}$$

$$(x_1, \dots x_n) \xrightarrow{\Phi} (x_1, \dots, x_{n-1}, x_n - \gamma(x_1, \dots x_{n-1}))$$

$$(y_1 \dots y_n)$$

$$\nu(x_0) \rightsquigarrow (0, \dots 0, 1)$$

$$(x_i) \qquad\qquad (y_i)$$

$\boxed{\partial U \cap B(x_0,r)}$ is the graph of γ

hypersurface in $\mathbb{R}^n$

$\downarrow$

"dS" area element on $\partial U \cap B(x_0,r)$

$\parallel$

"$\sqrt{1+|\nabla \gamma|^2}\, dx_1 \dots dx_{n-1}$"

$$\int_{\partial U \cap B(x_0,r)} f \dots dS$$

$f : \mathbb{R}^n \to \mathbb{R}$ $\qquad f(x_1 \dots x_n) \in \mathbb{R}$

∇f gradient of f $: \mathbb{R}^n \to \mathbb{R}^n$

$$\nabla f = (\partial_{x_1} f, \dots, \partial_{x_n} f) \qquad \cancel{Df} \quad ($$

$D^2 f$ hessian of f $\quad \mathbb{R}^n \to \mathbb{R}^{n \times n}$

$$D^2 f = \begin{pmatrix} \partial^2_{x_1 x_1} f & \partial^2_{x_1 x_2} f & \cdots \\ \vdots & \ddots & \vdots \\ \vdots & & \partial^2_{x_n x_n} f \end{pmatrix}$$

Δf = Laplacian of f $= tr(D^2 f)$

$$\Delta f(x) = \partial^2_{x_1 x_1} f(x) + \dots + \partial^2_{x_n x_n} f(x)$$

$$F : \mathbb{R}^n \to \mathbb{R}^n \qquad F(x) = (F^1(x_1 \dots x_n), \dots, F^n(x_1 \dots x_n))$$

$$\operatorname{div} F(x) = \partial_{x_1} F^1(x) + \partial_{x_2} F^2(x) + \dots + \partial_{x_n} F^n(x)$$

Obs. $\Delta f = \operatorname{div}(\nabla f)$

1) GAUSS - GREEN $\boxed{U \text{ bold open set of class } C^1}$

$$f \in C^1(\overline{U}) \qquad \int_U \partial_{x_i} f \, dx = \int_{\partial U} f(x) \underbrace{\nu_i(x)}_{\substack{i\text{-component}\\ \text{of the exterior normal } \nu.}} dS(x)$$

2) DIVERGENCE THEOREM

$$F : \mathbb{R}^n \to \mathbb{R}^n \qquad F \in \mathcal{C}^1(\overline{U})$$

ext normal

$$\int_U \operatorname{div} F \, dx = \int_{\partial U} \underbrace{F(x) \cdot \nu(x)}_{\text{the scalar product}} \, dS(x)$$

$$F_1 \cdot \nu_1 + F_2 \nu_2 + \ldots + F_n \nu_n$$

Conseq.

$f, g \in \mathcal{C}^2(\bar{U})$

scalar product

$$\int_U \Delta f \, dx = \int_{\partial \bar{U}} \nabla f(x) \cdot \nu(x) \, dS(x)$$

$$\underbrace{\int_U \Delta f \cdot g - f \cdot \Delta g \, dx}_{} = \int_{\partial U} f \cdot (\nabla g \cdot \nu) - g(\nabla f \cdot \nu) \, dS$$

Integration on spheres (particular case of CO-AREA formula)

(Folland, ch. 2, ~~§~~ sect 7)

$f : \mathbb{R}^n \to \mathbb{R}$ measurable, $\{ f \geq 0 \,/\, f \in L^1(\mathbb{R}^n)$

$$\boxed{\int_{\mathbb{R}^n} f(x)\,dx = \int_0^{+\infty} \int_{S^{n-1}} f(ry)\,dS(y) \cdot r^{n-1}\,dr \qquad S^{n-1} = \partial B(0,1)}$$

$$\int_0^{+\infty} \int_{\partial B(0,r)} f(z)\,dS(z)\,dr$$

obs if f is RADIAL $\quad f(x) = g(|x|)$

$$g : [0,+\infty) \longrightarrow \mathbb{R}$$

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^{+\infty} r^{n-1} g(r) \underbrace{A(\mathbb{S}^{n-1})}_{\text{area of the hypersurface } \mathbb{S}^{n-1} \,=\, \partial B(0,1)} dr$$

ex 1

$$f(x) = \begin{cases} \dfrac{1}{|x|^\alpha} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

$f \in L^1(\mathbb{R}^n) \quad (\Leftrightarrow f \in L^1(B(0,1)))$

$\Updownarrow$

$\alpha < n$

$$g(x) = \begin{cases} 0 & |x| \leq 1 \\ \dfrac{1}{|x|^\alpha} & |x| > 1 \end{cases}$$

$g(x) \in L^1(\mathbb{R}^n) \quad L^1(\mathbb{R}^n \setminus B(0,1))$

$\Updownarrow$

$\alpha > n$