Correction: $(\underbrace{\mathcal{C}(\bar{U})}_{\text{continuous functions in } U}, \|\cdot\|_\infty)$ is a closed subspace of $(\underbrace{L^\infty(U)}_{\text{essentially bdd functions.}}, \|\cdot\|_\infty)$

$$\|f\|_\infty = \sup_{x \in U} |f(x)|$$

$$C^{0,\gamma}(U) \subseteq \mathcal{C}(\bar{U}) \qquad \gamma \in (0,1]$$

$(C^{0,\gamma}(U), [\![\cdot]\!]_{C^{0,\gamma}})$ is a Banach space

$$[\![f]\!]_{C^{0,\gamma}} = \|f\|_\infty + \sup_{\substack{x \neq y \\ x,y \in U}} \frac{|f(x)-f(y)|}{|x-y|^\gamma}.$$

$U \subseteq \mathbb{R}^n$ open set (not nec. bdd; U can also be $\mathbb{R}^n$)

$\mathcal{C}_c^\infty(U) = \{ f : U \longrightarrow \mathbb{R}$ supp f is a compact contained in U (supp $f \subset\subset U$)

$D^\alpha f \in C(\bar{U}) \quad \forall\, \alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \}$

SPACE of TEST functions

put a topology on it (or just saying what does it mean "convergence")

$f_n \to f$ in $\mathcal{C}_c^\infty(U) \iff \exists\, K$ compact $K \subset\subset U$

supp f_n, supp $f \subseteq K$

$f_n \to f$ in $\mathcal{C}^\infty(K)$ $\left(\| D^\alpha f_n - D^\alpha f \|_\infty \to 0 \right.$ $\forall\, \alpha$

With this topology $\mathcal{C}^\infty_c(U)$ is a Hausdorff locally convex topological space

NOT Fréchet (just inductive limit of Fréchet spaces)

& NOT metrizable

(not Banach obv. since closed bdd sets are COMPACT)

CONVOLUTIONS

$p \in [1, +\infty)$ $L^p(U) = \{ f : U \to \mathbb{R}$ measurable and s. that $\int_U |f(x)|^p dx < +\infty \}$

$\|f\|_p = \left[\int_U |f(x)|^p \right]^{1/p}$ $\quad (L^p(U), \|\cdot\|_p)$ is BANACH

$p = +\infty$ $L^\infty(U) = \{ f : U \to \mathbb{R}$ measurable, $\exists c > 0$ $|f(x)| \le c$ a.e.$\}$

$\|f\|_\infty = \operatorname{ess\,sup}_{x \in U} |f(x)|$

product of convolution renders L^1 a Banach algebra

$f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ s. that for a.e x

$f(x-\cdot)\, g(\cdot) \in L^1(\mathbb{R}^n)$

$$\Rightarrow f * g(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$

YOUNG INEQUALITY let $f \in L^p(\mathbb{R}^n)$ $p \in [1, +\infty]$ $g \in L^q(\mathbb{R}^n)$ $q \in [1, +\infty]$

let $r \in [1, +\infty]$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

$\left(\frac{1}{\infty} = 0\right)$

then

$$f * g \in L^r(\mathbb{R}^n) \quad \|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

obs

$p = 1 = q \Rightarrow r = 1$ and PROVE by FUBINI TONELLI.

Prop 1) $\operatorname{supp} f * g \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$

2) $f \in L^1_{loc}(\mathbb{R}^n)$ that is $f \in L^1(K)$ $\forall K \subset\subset \mathbb{R}^n$ (compact)

$\phi \in C^\infty_c(\mathbb{R}^n) \Rightarrow f * \phi \in C^\infty(\mathbb{R}^n)$

3) $f \in C(\mathbb{R}^n)$ $\quad f * \phi \to f$ uniformly on compact sets

REGULARIZATION by CONVOLUTION

$$\rho(x) = \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x| \geq 1 \end{cases}$$

$\rho \in C^\infty_c(\mathbb{R}^n)$ $\quad \operatorname{supp} \rho = \overline{B(0,1)}$

$c \in \mathbb{R}$ chosen in order to have $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$

FREDERICKS MOLLIFIERS

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right) \quad \operatorname{supp} \rho_\varepsilon = \overline{B(0,\varepsilon)} \quad \rho_\varepsilon \in C^\infty_c(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} \rho_\varepsilon(x)\,dx = 1$$

<u>Prop</u> $f \in L^p(\mathbb{R}^n)$ $\quad \rho_\varepsilon * f \longrightarrow f$ in $L^p(\mathbb{R}^n)$ $\quad \forall p \in [1, +\infty)$

$$\|\rho_\varepsilon * f\|_{L^p} \leq \|f\|_p \qquad \text{(YOUNG)}$$

$+\infty$ excluded (why?)

CONSEQUENCES

1) $\overline{C_c^\infty(U)}^{\|\cdot\|_{L^p}} = L^p(U) \quad \forall U \subseteq \mathbb{R}^n$ open.

sketch: (take $U = \bigcup_{k=1}^{\infty} U_k \qquad U_{k+1} \supseteq \overline{U_k} \qquad \operatorname{dist}(\overline{U_k}, \mathbb{R}^n \setminus U) > \frac{2}{k}$

$(f\chi_{U_k} * \rho_{1/k}) \in C_c^\infty(U)$, $\; f\chi_{U_k} * \rho_{1/k} \to f$ in $L^p(U)$.

2) LEMMA DUBOIS-REYMOND
(also called fundam. lemma calculus of variations).

$U \subseteq \mathbb{R}^n$ open $\quad f \in L^1_{loc}(U)$.

If $\int_U f(x)\phi(x)dx = 0 \quad \forall \phi \in C_c^\infty(U)$ then $f = 0$ a.e.

proof Fix $K \subset\subset U$

$$g(x) = \begin{cases} \text{sign} f(x) \left(= \frac{f(x)}{|f(x)|} \right) & x \in K \\ 0 & x \in U \setminus K \end{cases}$$

(NB $\text{sign} f(x) = \begin{cases} \frac{f(x)}{|f(x)|} & \text{if } f(x) \neq 0 \\ 0 & f(x) = 0 \end{cases}$

$\varepsilon \to 0$

$g * \rho_\varepsilon \to g$ in $L^1(U)$ $\qquad g * \rho_\varepsilon \in C_c^\infty(U)$ if ε

is suff. small $\left(\varepsilon < \underbrace{2\,\mathrm{dist}(K, \mathbb{R}^n \setminus U)}_{=\varepsilon_0} . \right)$ since

$$\mathrm{supp}\,(g * \rho_\varepsilon) \subseteq K + B(0, \varepsilon)$$

So by ass. $\displaystyle\int_U (g * \rho_\varepsilon) \cdot f(x) = 0$ for $\varepsilon \leq \varepsilon_0$

$\downarrow$ sending $\varepsilon \to 0$

$\displaystyle\int_{K+B(0,\varepsilon_0)} |f(x)| = 0 \quad \forall K \Rightarrow f = 0$ a.e. in K

LEMMA
(USED TO PROVE UNIQUENESS ...)

$\Rightarrow f = 0$ a.e in U

1-dim variant of the lemma:

Let $f \in L^1_{loc}(a,b)$ such that $\displaystyle\int_a^b f(x)\,\phi'(x)\,dx = 0 \quad \forall \phi \in C_c^\infty(a,b)$

then $\exists\, c \in \mathbb{R}$ s. that $f(x) = c$ a.e.

proof fix a generic $w \in C_c^\infty(a,b)$

and consider $\psi \in C_c^\infty(a,b)$ $\int_a^b \psi(x)\,dx = 1$

then $\phi(x) = \int_a^x w(t)\,dt - \left[\int_a^b w(s)\,ds\right] \int_a^x \psi(t)\,dt$

satisfies
$$\begin{cases} \phi(a) = 0 = \phi(b) \\ \phi'(x) = w(x) - \left[\int_a^b w(s)\,ds\right] \psi(x) \in C_c^\infty(a,b) \\ \phi \in C_c^\infty(a,b) \end{cases}$$

by assumption

$$0 = \int_a^b f(x)\,\phi'(x)\,dx = \int_a^b f(x)w(x)\,dx - \left(\int_a^b w(s)\,ds\right) \int_a^b f(x)\psi(x)\,dx$$

$$= \int_a^b f(x)\,w(x)\,dx - \left[\int_a^b w(x)\,dx\right] \int_a^b f(s)\,\psi(s)\,ds =$$

$$= \int_a^b \left\{ f(x)\, w(x) - w(x) \left[\int_a^b f(s)\,\psi(s)\,ds \right] \right\} dx =$$

$$= \int_a^b w(x) \left[f(x) - \int_a^b f(s)\,\psi(s)\,ds \right] dx = 0$$

true $\forall\, w \in C_c^\infty(a,b)$ by DB-R lemma $\rightarrow$

$$f(x) - \int_a^b f(s)\,\psi(s)\,ds = 0 \quad \text{a.e.} \qquad \square.$$

<u>Prop</u> Let $K \subset\subset U$ $\quad U$ open set.

$\exists\, \phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi(x) \equiv 1 \quad \forall x \in K$

$0 \le \phi(x) \le 1 \quad \forall x \in \mathbb{R}^n$

$\operatorname{supp} \phi \subset\subset U \quad$ (so $\phi \equiv 0$ in $\mathbb{R}^n \setminus U$)

proof (sketch:) $\quad \delta = \operatorname{dist}(K, \mathbb{R}^n \setminus U) > 0$

take $\varepsilon', \varepsilon > 0 \quad \varepsilon' + \varepsilon < \delta \quad \varepsilon < \varepsilon'$

Let $f(x) = \begin{cases} 1 & \operatorname{dist}(x, K) \leq \varepsilon' \\ 0 & \text{elsewhere} \end{cases}$

$(\operatorname{supp} f = K + B(0, \varepsilon'))$

$f * \rho_\varepsilon \in C_c^\infty(U) \subseteq C_c^\infty(\mathbb{R}^n) \qquad \operatorname{supp}(f * \rho_\varepsilon) \subseteq K + B(0,\varepsilon) + B(0,\varepsilon') \subset\subset U$

$0 \leq f * \rho_\varepsilon(x) \leq 1 \quad \forall x \in \mathbb{R}^n$

$x \in U$ such that $d(x, K) \leq \overbrace{\varepsilon' - \varepsilon}^{>0!}$

$$f * \rho_\varepsilon(x) = \int_{\mathbb{R}^n} f(y) \rho_\varepsilon(x-y)\, dy = \int_{B(x,\varepsilon)} f(y) \rho_\varepsilon(x-y)\, dy = \int_{B(x,\varepsilon)} 1 \cdot \rho_\varepsilon(x-y)\, dy = 1$$

if $y \in B(x, \varepsilon)$, $\operatorname{dist}(y, K) \leq \varepsilon + \underbrace{\operatorname{dist}(x, K)}_{\leq \varepsilon'}$