Lesson october 16 will be cancelled.

The course will finish middle of December
(16, 17 December)

PRELIMINARIES

- Function spaces

$U$ open bdd set in $\mathbb{R}^n$

$f: U \to \mathbb{R}$ $\gamma$- Hölder continuous in $U$ if

$\gamma \in (0,1)$

$$C = \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|^{\gamma}} < +\infty \Rightarrow f \in C(\overline{U})$$

$f \in \mathcal{C}^{0,\gamma}(U)$ = set of $\gamma$-Hölder cont. functions

$f \in \mathcal{C}^{0,\gamma}_{loc}(U)$ ( locally- $\gamma$-Holder cont. in $U$)

if $\forall K$ compact set in $U$

$$C_K = \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|^\gamma} < +\infty$$

$f \in \mathcal{C}^{0,1}(U)$ Lipschitz continuous functions

$f \in \mathcal{C}^{0,1}_{loc}(U)$ locally Lipschitz functions

**obs** given $f \in C^{0,1}(U) \longrightarrow \exists \bar{f}$ extension of $f$ such that $\bar{f} \in C^{0,1}(\mathbb{R}^n)$

$\bar{f}(x) = f(x) \quad \forall x \in \bar{U}$

$$\sup_{\substack{x \neq y \\ x,y \in U}} \frac{|f(x) - f(y)|}{|x-y|} = C < +\infty$$

$$\sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|\bar{f}(x) - \bar{f}(y)|}{|x-y|} = C$$

$$\bar{f}_1(x) = \sup_{y \in U} \left[ f(y) - C\,|x-y| \right]$$ largest extension

$$\bar{f}_2(x) = \inf_{y \in U} \left[ f(y) + C\,|x-y| \right]$$ smallest extension.

$\mathcal{C}^{0,\gamma}(U)$ $\quad \gamma \in (0,1]$

$f \in \mathcal{C}^{0,\gamma}(U)$ $\quad C_f =$ Hölder constant of $f$ (if $\gamma < 1$)

(Lipschitz constant of $f$ if $\gamma = 1$)

$$= \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma}$$

$$[\![f]\!]_{\mathcal{C}^{0,\gamma}(U)} = \underbrace{\|f\|_\infty}_{\sup_{x \in U} |f(x)|} + C_f$$

$C_f \to$ Hölder constant

$$\left(\mathcal{C}^{0,r}(U), [\![\cdot]\!]_{\mathcal{C}^{0,r}}\right) \subseteq \left(L^\infty(U), \|\cdot\|_\infty\right)$$

$k \in \mathbb{N}$ $\quad r \in [0,1]$

$|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \leq k$

$\alpha = (\alpha_1 \ldots \alpha_n)$

$$\mathcal{C}^{k,r}(U) = \Big\{ f : U \to \mathbb{R} \quad \underbrace{D^\alpha f \in \mathcal{C}^{0,r}(U)} \Big\}$$

$$\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n} f$$

# Ascoli - Arzelà compactness theorem

$\underbrace{f_n \in C(\overline{U})}$ $\quad U$ bdd open set ($\overline{U}$ compact)

$f_n$ is a seq. of continuous f. in $\overline{U}$

- $\|f_n\|_\infty \leq C$ (UNIF bounded, C is independ. of $n$)
- $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$$|x-y| \leq \delta \quad x, y \in \overline{U} \Rightarrow \underbrace{|f_n(x) - f_n(y)| \leq \varepsilon}_{\forall n \in}$$

EQUICONTINUOUS

$\Rightarrow$ UP TO PASSING TO A SUBSEQUENCE $f_{n_k}$

$f_{n_k} \longrightarrow f \in \mathcal{C}(\bar{U})$ uniformly

$(\|f_{n_k} - f\|_\infty \to 0)$

If $X$ is $\sigma$-compact $\left[ X = \bigcup_n \bar{U}_n \right.$ $U_n$ open bdd sets

$\left. \bar{U}_n \subseteq U_{n+1} \right]$

$f_n$ is a seq. of functions which are equibdd and equicontinuous in $X$ $\Rightarrow$

$f_{n_k} \to f \in C(X)$ locally uniformly $\|$

$\forall K \subset X$ $K$ compact $\|f_{n_k} - f\|_{L^\infty(K)} \to 0$

$U$ open bdd

**Obs** $f_n \in C^{0,\gamma}(U)$ $[f_n]_{C^{0,\gamma}(U)} \le \underline{\underline{C}}$

(seq. of $\gamma$-Hölder fun. with uniformly bdd norm)

$\exists$ a subseq. $f_{n_k}$, $\exists f \in C^{0,\gamma}(U)$

such that

$$[f_{n_k} - f]_{C^{0,\beta}(U)} \to 0 \quad \forall \beta < \gamma$$

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$\forall \beta < \gamma$ $C^{0,\gamma}(U) \hookrightarrow C^{0,\beta}(U)$ is COMPACT

**Def** $f \in C(U)$

$$\operatorname{supp} f = \overline{\{x \in U, \ f(x) \neq 0\}}$$

SUPPORT of $f$

more generally

$$\operatorname{supp} f = U \setminus A$$

$$A = \bigcup \{\text{open sets } B \subseteq U \quad f = 0 \text{ a.e in } B\}$$

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**Def** $C_c(U)$ = function $f : U \to \mathbb{R}$ continuous in $U$, with $\underbrace{\operatorname{supp} f}_{} \subset\subset U$

supp $f$ is a compact set inside $U$.

$$\overline{\mathcal{C}_c(U)}^{\|\cdot\|_\infty} = \mathcal{C}_0(U)$$ (functions which are 0 on $\partial U$).

$$= \left\{ f : U \to \mathbb{R} \quad f \text{ continuous}, \quad \forall \varepsilon > 0 \quad \{ f(x) \geq \varepsilon \} \text{ compact inside } U \right\}$$

Take $U \subseteq \mathbb{R}^n$ OPEN SET (not necessarily BOUNDED)

$K$ compact set $K \subset U$

$k \in \mathbb{N}$

$$\mathcal{C}^k(K) = \{ f : K \to \mathbb{R} \quad D^\alpha f \in \mathcal{C}(K) \quad \forall \alpha \ (\alpha_1 + \alpha_2 + \dots + \alpha_n \leq k) \}$$

$$\|f\|_{\mathcal{C}^k} = \sup_{\substack{\alpha \\ 0 \leq \alpha_1 + \alpha_2 + \dots + \alpha_n \leq k}} \|D^\alpha f\|_{L^\infty(K)}$$

$(\mathcal{C}^k(K), \|\cdot\|_k)$ is Banach

$$\mathcal{C}^\infty(K) = \{ D^\alpha f \in \mathcal{C}(K) \quad \forall \alpha = (\alpha_1, \dots, \alpha_n) \}$$

$\| \cdot \|_{\mathcal{C}^k} \quad \forall k \in \mathbb{N}$

$f_n \longrightarrow f$ in $\mathcal{C}^\infty(K)$

if $\|f_n - f\|_{\mathcal{C}^k} \longrightarrow 0 \quad \forall k \in \mathbb{N}$

$$d(f, g) = \sum_{k \in \mathbb{N}} \frac{2^{-k} \|f - g\|_{\mathcal{C}^k}}{(1 + \|f - g\|_{\mathcal{C}^k})}$$

(closed bdd sets are compact by A.A.)

(complete metric space (NOT Banach space!))

FRECHET SPACE