

Sheet 1 (Preliminaries and measure theory)

Ex 1 (1) $C^{0,\alpha}(\bar{U}) \subseteq L^\infty(U)$ as a subspace. Let u_n be a Cauchy sequence in $C^{0,\alpha}(\bar{U})$. Then u_n is a Cauchy sequence in $L^\infty(U)$, and there $\exists u \in C(\bar{U})$ such that $u_n \rightarrow u$ uniformly (in L^∞ norm)

(note that convergence in L^∞ is for continuous functions the uniform convergence).

By pointwise convergence $\forall x, y \in \bar{U}$

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq \liminf_n \frac{|u_n(x) - u_n(y)|}{|x - y|^\alpha} \leq \|u_n\|_{C^{0,\alpha}(\bar{U})} \leq C$$

since u_n is bounded in $C^{0,\alpha}(\bar{U})$
 \downarrow

so $u \in C^{0,\alpha}(\bar{U})$.

Finally

$$\frac{|(u - u_n)(x) - (u - u_n)(y)|}{|x - y|^\alpha} \leq \liminf_k \frac{|(u_k - u_n)(x) - (u_k - u_n)(y)|}{|x - y|^\alpha} \rightarrow 0$$

so $\|u_n - u\|_{C^{0,\alpha}(\bar{U})} \rightarrow 0$ as $n \rightarrow \infty$

(2) First of all we get that $\|u_n\|_\infty \leq C$ and moreover
 $\|u_n(x) - u_n(y)\| \leq C|x-y|^\alpha \quad \forall x, y \in \bar{U} \Rightarrow$ by A.A. up to
 subsequences $u_n \rightarrow u$ in $C(\bar{U})$. By the same argument
 as in (1), $u \in C^{0,\alpha}(\bar{U})$. $\|u\|_{C^{0,\alpha}} \leq C$.

Let $\beta < \alpha$. Fix $\delta > 0$ and consider

$$\textcircled{1} \quad |x-y| \leq \delta \quad \frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} = |x-y|^{\alpha-\beta} 2 \|u_n - u\|_{C^{0,\alpha}(\bar{U})} \\ \leq \delta^{\alpha-\beta} 2C$$

$$\textcircled{2} \quad |x-y| \geq \delta \quad \frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} \leq \delta^{-\beta} 2 \|u - u_n\|_\infty$$

so for each $\delta > 0$ consider $n \geq n(\delta)$ $\|u - u_n\|_\infty \leq \delta^\alpha$

and we get

$$\frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} \leq \overline{C} \delta^{\alpha-\beta} \xrightarrow{\delta \rightarrow 0} 0.$$

Ex 2

(1)

Note that $\text{div}(xu^2) = nu^2 + 2u x \cdot \nabla u$

$$\textcircled{*} \int_{B(0,r)} nu^2 + 2u x \cdot \nabla u = \int_{\partial B(0,r)} u^2 x \cdot \frac{x}{|x|} dS = r \int_{\partial B(0,r)} u^2 dS$$

since
 $\nabla(x) = \frac{x}{|x|}$
 on $\partial B(0,r)$

Note that

$$|2u x \cdot \nabla u| \leq 2|u| |x| \cdot |\nabla u| \leq 2|u|r \cdot |\nabla u| \leq |u|^2 + r^2 |\nabla u|^2$$

Substitute in $\textcircled{*}$ and divide by r^2 .

(2) first equality is just divergence theorem. Observe that

$$|2u \frac{x \cdot \nabla u}{|x|^2}| \leq 2 \frac{|u|}{|x|} |\nabla u| \leq \frac{1}{\delta} |\nabla u|^2 + \delta \frac{u^2}{|x|^2} \quad \text{for every } \delta > 0.$$

(3) By (1)

$$\int_{B_r \setminus B_\varepsilon} (n-2-\delta) \frac{u^2}{|x|^2} \leq \frac{1}{\delta} \int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \frac{(n+1)}{r^2} + \int_{B_r} |\nabla u|^2 dx - \underbrace{\frac{1}{\varepsilon} \int_{\partial B_\varepsilon} u^2 dS}_{\leq \|u\|_\infty^2 \cdot \frac{|\partial B_\varepsilon|}{\varepsilon} = \|u\|_\infty^2 \cdot C_\varepsilon^{n-2} \rightarrow 0} \quad \forall \varepsilon < r$$

So $\lim_{\varepsilon \rightarrow 0^+} \int_{B_r \setminus B_\varepsilon} (n-2-\delta) \frac{u^2}{|x|^2} dx$ exists finite.

(4) $u(0) \neq 0 \Rightarrow$ assume without loss of generality $u(0) = \alpha > 0$.

Here $\exists \varepsilon' \quad u(x) \geq \frac{\alpha}{2} \quad \forall x \in B_{\varepsilon'}$

by integration on spheres (co-area f.)

$$\int_{\mathbb{R}^n} \frac{u^2(x)}{|x|^2} dx \geq \int_{B_{\varepsilon'}} \frac{u^2(x)}{|x|^2} dx \geq \frac{\alpha^2}{4} \int_{B_{\varepsilon'}} \frac{1}{|x|^2} dx = \frac{\alpha^2}{4} \int_0^{\varepsilon'} \frac{m \omega_m r^{m-1}}{r^2} = \frac{\alpha^2}{4} m \omega_m \int_0^{\varepsilon'} r^{m-3} dr$$

\downarrow
 $+\infty$
 if $m=1, m=2$

If $u, |\nabla u| \in L^2(\mathbb{R}^n)$ choose $\delta = \frac{n-2}{2}$ in (3) and

we get

$$\frac{n-2}{2} \int_{B_r} \frac{u^2(x)}{|x|^2} dx \leq \frac{2}{n-2} \int_{B_r} |\nabla u|^2 + \underbrace{\frac{1}{r} \int_{\partial B_r} u^2 dS}_{\downarrow}$$

sending $r \rightarrow +\infty$ we conclude.

0 as $r \rightarrow +\infty$

$$\int_{\mathbb{R}^n} u^2 = \int_0^{+\infty} \int_{\partial B_r} u^2 dS < +\infty$$

so limiting $r \rightarrow +\infty$

$$\int_{\partial B_r} u^2 dS = 0$$