Preliminaries on calculus

Ex 1) 1) $(\mathcal{C}(\bar{U}), \|\cdot\|_\infty)$ is a closed subspace of $(L^\infty(U), \|\cdot\|_\infty)$

since $\mathcal{C}(\bar{U}) \subseteq L^\infty(U)$ and $u_n \in \mathcal{C}(\bar{U})$ Cauchy in L^∞

$\Rightarrow u_n \to u$ uniformly in $\mathcal{C}(\bar{U}) \Rightarrow u \in \mathcal{C}(\bar{U})$

Let u_n be a Cauchy seq. in $\mathcal{C}^{0,\alpha}(U) \Rightarrow u_n$ is Cauchy in $\mathcal{C}(\bar{U}) \Rightarrow u_n \to u$ uniformly in $\bar{U}$

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \underset{\text{by pointwise limit}}{\leq} \lim_n \frac{|u_n(x)-u_n(y)|}{|x-y|^\alpha} \leq C \Rightarrow u \in \mathcal{C}^{0,\alpha}(U)$$

Finally
$$\frac{|(u-u_n)(x)-(u-u_n)(y)|}{|x-y|^\alpha} \leq \lim_k \underbrace{\frac{|(u_k-u_n)(x)-(u_k-u_n)(y)|}{|x-y|^\alpha}}_{\to 0 \text{ as } k,n\to+\infty}$$

$u_n \to u$ in $\mathcal{C}^{0,\alpha}(\bar{U})$.

(2) First of all we get that $\|u_n\|_\infty \leq C$ and moreover

$|u_n(x) - u_n(y)| \leq C|x-y|^\alpha \quad \forall x,y \in \bar{U} \Rightarrow$ by A.A. up to

subsequences $u_n \to u$ in $\mathcal{C}(\bar{U})$. By the same argument

as in (1), $u \in \mathcal{C}^{0,\alpha}(\bar{U})$. $\|u\|_{\mathcal{C}^{0,\alpha}} \leq C$.

Let $\beta < \alpha$. Fix $\delta > 0$ and consider

① $|x-y| \leq \delta$

$$\frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} = |x-y|^{\alpha-\beta} 2\|u_n - u\|_{\mathcal{C}^{0,\alpha}(\bar{U})}$$

$$\leq \delta^{\alpha-\beta} 4C$$

② $|x-y| \geq \delta$

$$\frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} \leq \delta^{-\beta} 2\|u-u_n\|_\infty$$

So for each $\delta > 0$ consider $n \geq n(\delta)$ $\|u-u_n\|_\infty \leq \delta^\alpha$

and we get

$$\frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} \leq \tilde{C}\delta^{\alpha-\beta} \xrightarrow[\delta \to 0]{} 0.$$

Ex 2

(1) Note that $\operatorname{div}(xu^2) = nu^2 + 2u\, x\cdot\nabla u$

$$(*)\quad \int_{B(0,r)} nu^2 + 2u\, x\cdot\nabla u = \int_{\partial B(0,r)} u^2 x\cdot\frac{x}{|x|}\, dS = r\int_{\partial B(0,r)} u^2\, dS$$

since $\nu(x) = \frac{x}{|x|}$ on $\partial B(0,r)$

Note that

$$|2u\, x\cdot\nabla u| \le 2|u|\,|x|\cdot|\nabla u| \le 2|u| r\cdot|\nabla u| \le |u|^2 + r^2|\nabla u|^2$$

substitute in $(*)$ and divide by r^2.

(2) first equality is just divergence theorem. Observe that

$$\left|2u\frac{x}{|x|^2}\cdot\nabla u\right| \le 2\frac{|u|}{|x|}|\nabla u| \le \frac{1}{\delta}|\nabla u|^2 + \delta\frac{u^2}{|x|^2}.\quad \text{for every } \delta>0.$$

(3) By (1)

$$\int_{B_r\setminus B_\varepsilon}(n-2-\delta)\frac{u^2}{|x|^2} \le \frac{1}{\delta}\int_{B_r}|\nabla u|^2 + \int_{B_r} u^2\frac{(n+1)}{r^2} + \int_{B_r}|\nabla u|^2\, dx \underbrace{- \frac{1}{\varepsilon}\int_{\partial B_\varepsilon} u^2\, dS}_{\le \|u\|_\infty^2\cdot\frac{|\partial B_\varepsilon|}{\varepsilon} = \|u\|_\infty^2\cdot C\varepsilon^{n-2} \to 0} \quad \forall\varepsilon<r$$

So $\lim_{\varepsilon\to 0^+}\int_{B_r\setminus B_\varepsilon}(n-2-\delta)\frac{u^2}{|x|^2}\, dx$ exists finite.

(4) $u(0)\neq 0 \Rightarrow$ assume without loss of generality $u(0)=\alpha>0$.

then $\exists\, \varepsilon'$ $\quad u(x) \geq \frac{\alpha}{2} \quad \forall x \in B_{\varepsilon'}$

by integration on spheres (co-area f.)

$$\int_{B_r} \frac{u^2(x)}{|x|^2}\,dx \geq \int_{B_{\varepsilon'}} \frac{u^2(x)}{|x|^2}\,dx \geq \frac{\alpha^2}{4} \int_{B_{\varepsilon'}} \frac{1}{|x|^2}\,dx = \frac{\alpha^2}{4} \int_0^{\varepsilon'} \frac{n\omega_n r^{n-1}}{r^2} = \frac{\alpha^2}{4}\, n\omega_n \underbrace{\int_0^{\varepsilon'} r^{n-3}\,dr}_{+\infty}$$

if $n=1$, $n=2$

If $u, |\nabla u| \in L^2(\mathbb{R}^n)$ choose $\delta = \frac{n-2}{2}$ in (3) and we get

$$\frac{n-2}{2} \int_{B_r} \frac{u^2(x)}{|x|^2}\,dx \leq \frac{2}{n-2} \int_{B_r} |\nabla u|^2 + \underbrace{\frac{1}{r} \int_{\partial B_r} u^2\,dS}_{\to\, 0 \text{ as } r \to +\infty}$$

sending $r \to +\infty$ we conclude.

$$\int_{\mathbb{R}^n} u^2 = \int_0^{+\infty} \int_{\partial B_r} u^2\,dS < +\infty$$

so $\liminf\limits_{r \to +\infty} \int_{\partial B_r} u^2\,dS = 0$