LESSON 21: SALT-WEDGE INTRUSION

ICEA

THE SPEED OF THE INTERFACE WAVE

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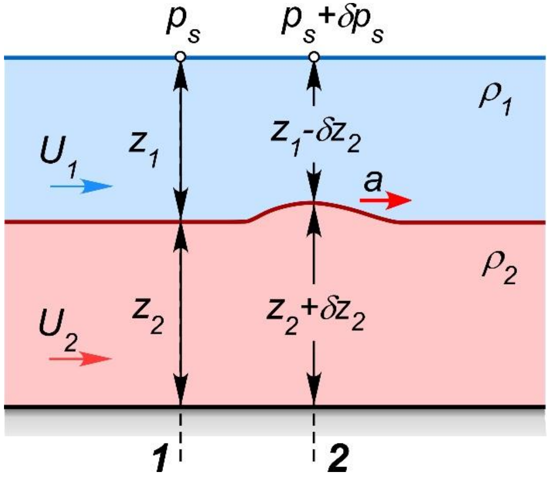
Let us analyze the speed of a small perturbation a along the fluid interface on assuming the rigid-lid approximation.



For the observer moving with velocity a, the continuity results:

$$(U_1 - a)z_1 = (U_1 + \delta U_1 - a)(z_1 - \delta z_2) \qquad 1$$

$$(U_2 - a)z_2 = (U_2 - \delta U_2 - a)(z_2 + \delta z_2) \qquad 2$$

On assuming the energy conservation between 1 and 2, we find for the layer 1:

$$\cancel{\frac{p_s}{g\rho_1}} + \cancel{z_1} + \cancel{z_2} + \frac{(U_1 - a)^2}{2g} = \frac{\cancel{p_s} + \delta p_s}{g\rho_1} + \cancel{z_1} + \cancel{z_2} + \frac{(U_1 + \delta U_1 - a)^2}{2g}$$

$$\longrightarrow \frac{(U_1 - a)^2}{2g} = \frac{\delta p_s}{g\rho_1} + \frac{(U_1 + \delta U_1 - a)^2}{2g} \qquad 3$$

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Similarly, for the layer 2:

$$\frac{\cancel{p_s}}{\cancel{g}\rho_2}+\cancel{z_2}+\frac{\rho_1}{\cancel{\rho_2}}z_1+\frac{(U_2-a)^2}{2g}=\frac{\cancel{p_s}+\delta p_s}{g\rho_2}+\cancel{z_2}+\delta z_2+\frac{\rho_1}{\rho_2}(\cancel{z_1}-\delta z_2)+\frac{(U_2-\delta U_2-a)^2}{2g}$$

$$\longrightarrow \quad \frac{(U_2-a)^2}{2g}=\frac{\delta p_s}{g\rho_2}+\delta z_2-\frac{\rho_1}{\rho_2}\delta z_2+\frac{(U_2-\delta U_2-a)^2}{2g} \qquad 4$$

By combining (1) and (3) in order to eliminate the kinetic term of the section 2, we have:

$$\frac{(U_1-a)^2}{2\cancel{g}}=\frac{\delta p_s}{\cancel{g}\rho_1}+\frac{{z_1}^2}{2\cancel{g}}\frac{(U_1-a)^2}{(z_1-\delta z_2)^2}$$

$$(U_1+\delta U_1-a)=z_1\frac{(U_1-a)}{(z_1-\delta z_1)}$$

$$\frac{\delta p_s}{\rho_1}=\frac{(U_1-a)^2}{2}-\frac{{z_1}^2}{2}\frac{(U_1-a)^2}{(z_1-\delta z_2)^2}$$

$$\longrightarrow \quad \delta p_s=\rho_1\frac{(U_1-a)^2}{2}\left[1-\frac{{z_1}^2}{(z_1-\delta z_2)^2}\right] \qquad 5$$

On the other hand, eqs (2) and (4) yields:

$$\frac{(U_2-a)^2}{2g} = \frac{\delta p_s}{g\rho_2} + \delta z_2 - \frac{\rho_1}{\rho_2}\delta z_2 + \frac{{z_2}^2}{2g}\frac{(U_2-a)^2}{(z_2+\delta z_2)^2}$$

$$(U_2-\delta U_2-a) = z_2\frac{(U_2-a)}{(z_2+\delta z_1)}$$

$$\frac{(U_2-a)^2}{2g}\left[1-\frac{{z_2}^2}{(z_2+\delta z_2)^2}\right] = \frac{\delta p_s}{g\rho_2} + \frac{\Delta\rho}{\rho_2}\delta z_2$$

$$\delta z_2 - \frac{\rho_1}{\rho_2}\delta z_2 = \frac{\rho_2-\rho_1}{\rho_2}\delta z_2 = \frac{\Delta\rho}{\rho_2}\delta z_2$$

$$\frac{(U_2-a)^2}{2g}\left[1-\frac{{z_2}^2}{(z_2+\delta z_2)^2}\right] = \frac{\rho_1}{\rho_2}\frac{(U_1-a)^2}{2g}\left[1-\frac{{z_1}^2}{(z_1-\delta z_2)^2}\right] + \frac{\Delta\rho}{\rho_2}\delta z_2$$

By (5)

$$\frac{(U_2-a)^2}{2g}\left[\frac{2z_2\delta \cancel{z_2}+\delta {z_2}^{\cancel{2}}}{(z_2+\delta z_2)^2}\right] = \frac{\rho_1}{\rho_2}\frac{(U_1-a)^2}{2g}\left[\frac{-2z_1\delta \cancel{z_2}+\delta {z_2}^{\cancel{2}}}{(z_1-\delta z_2)^2}\right] + \frac{\Delta\rho}{\rho_2}\delta \cancel{z_2}$$

$$\frac{(U_2-a)^2}{2g}\left[\frac{2z_2+\delta z_2}{(z_2+\delta z_2)^2}\right] = \frac{\rho_1}{\rho_2}\frac{(U_1-a)^2}{2g}\left[\frac{-2z_1+\delta z_2}{(z_1-\delta z_2)^2}\right] + \frac{\Delta\rho}{\rho_2}$$

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On assuming small density difference between the two layers and $\delta z_2 \rightarrow 0$, the latter reads:

$$\frac{(U_2-a)^2}{2g}\frac{2z_2}{{z_2}^2} = -\cancelto{1}{\frac{\rho_1}{\rho_2}}\frac{(U_1-a)^2}{2g}\frac{2z_1}{{z_1}^2} + \frac{\Delta\rho}{\rho}$$

$$\longrightarrow \qquad \frac{(U_2-a)^2}{z_2} + \frac{(U_1-a)^2}{z_1} = g\frac{\Delta\rho}{\rho} = g'$$

This relatioship can be used to evaluate the speed a. Its expansion indeed reads:

$$z_1(U_2-a)^2 + z_2(U_1-a)^2 = g' z_1 z_2$$

$$z_1{U_2}^2 + z_1 a^2 - 2z_1U_2a + z_2{U_1}^2 + z_2a^2 - 2z_2U_1a = g'z_1z_2$$

$$a^2(z_1+z_2) - 2a(z_1U_2 + z_2U_1) + z_1{U_2}^2 + z_2{U_1}^2 - g'z_1z_2 = 0$$

The roots of this second degree equation are finally:

$$a = \frac{z_1U_2 + z_2U_1}{z_1+z_2} \pm \sqrt{\frac{(z_1U_2+z_2U_1)^2 - (z_1+z_2)\left(z_1{U_2}^2 + z_2{U_1}^2 - g'z_1z_2\right)}{(z_1+z_2)^2}}$$

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By expanding the terms into the square-root, the wave speed results:

$$a = \frac{z_1 U_2 + z_2 U_1}{z_1 + z_2} \pm \sqrt{g' \frac{z_1 z_2}{z_1 + z_2}\left[1 - \frac{(U_1 - U_2)^2}{g'(z_1 + z_2)}\right]}$$

Hence, the solutions exist if the argument into the square-root is positive, namely:

$$\frac{(U_1 - U_2)^2}{g'(z_1 + z_2)} < 1$$

It is a Richardson number and represents the conditon of stable stratification

$$a = \frac{z_1 U_2 + z_2 U_1}{z_1 + z_2} \pm \sqrt{\frac{\cancel{{z_1}^2 {U_2}^2} + \cancel{{z_2}^2 {U_1}^2} + 2 z_1 z_2 U_2 U_1 - \cancel{{z_1}^2 {U_2}^2} - z_1 z_2 {U_2}^2 - z_1 z_2 {U_1}^2 - \cancel{{z_2}^2 {U_1}^2} + g' {z_1}^2 z_2 + g' z_1 {z_2}^2}{(z_1 + z_2)^2}}$$

$$a = \frac{z_1 U_2 + z_2 U_1}{z_1 + z_2} \pm \sqrt{\frac{2 z_1 z_2 U_2 U_1 - z_1 z_2 {U_2}^2 - z_1 z_2 {U_1}^2 + g' {z_1}^2 z_2 + g' z_1 {z_2}^2}{(z_1 + z_2)^2}}$$

$$a = \frac{z_1 U_2 + z_2 U_1}{z_1 + z_2} \pm \sqrt{\frac{g' z_1 z_2 (z_1 + z_2) - z_1 z_2 \left({U_2}^2 - 2 U_2 U_1 + {U_1}^2\right)}{(z_1 + z_2)^2}}$$

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It is worth noting that for $U_1 = 0$ and $z_1 \gg 1$ the problem reduces in:

$$a = U_2 \pm \sqrt{g' z_2}$$

and assuming the case air-water $g' \cong g$, the relationship is the same of the standard solution for homogeneous fluid.

$$a = \frac{U_2 + \cancelto{0}{U_1 z_2/z_1}}{\cancelto{1}{z_1/z_1} + \cancelto{0}{z_2/z_1}} \pm \sqrt{g' \frac{z_2}{\cancelto{1}{z_1/z_1} + \cancelto{0}{z_2/z_1}} \left[1 - \cancelto{0}{\frac{(U_1 - U_2)^2}{g'(z_1 + z_2)}}\right]}$$

THE CRITICAL CONDITION

The notions of subcritical and supercritical flow, as well as the concept of critical depth are valid also in the stratified flow. However, there are some dissimilarities with respect to the homogeneous flow.

In particular, the flow is defined supercritical, if $a > 0$ disregarding the sign of the roots, vice versa, in the subcritcal condition one of the roots is negative. Similarly to the homogenous fluid, the case $a = 0$ identifies the critical condition of the flow.

From the mathematical point of view:

$$\frac{(U_2 - a)^2}{z_2} + \frac{(U_1 - a)^2}{z_1} = g' \longrightarrow \frac{{U_2}^2}{z_2} + \frac{{U_1}^2}{z_1} = g'$$

Namely:

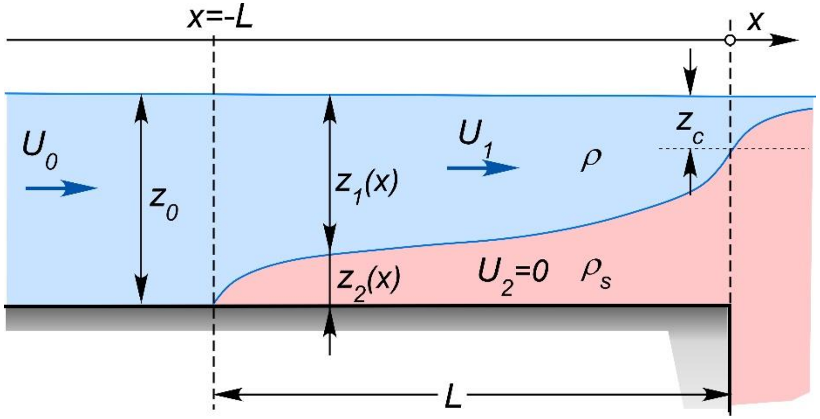
$$\frac{{U_2}^2}{g' z_2} + \frac{{U_1}^2}{g' z_1} = 1 \longrightarrow {F_1}'^2 + {F_2}'^2 = G'^2 = 1$$

Composite Froude number

This solution depends on the rigid-lid approximation. By removing this hypothesis, the critical condition results:

$$G'^2 = {F_1}'^2 + {F_2}'^2 - \frac{\Delta\rho}{\rho} {F_1}'^2 {F_2}'^2 = 1$$

The salt wedge is the upstream intrusion of the seawater along the estuarine of the rivers. Since there is a difference of density between salt water and fresh water, the phenomenon occurs as stratified flow, as shown in the figure below.



Assumptions:

- Steady flow
- Immiscible fluid (ρ freshwater, ρ_s seawater)
- Hydrostatic pressure distribution in z
- Horizontal free surface

$\longrightarrow$ $U_2 = 0$

The solution of the problems is given by the 1D flow equation of each layer, namely:

$$\frac{\partial}{\partial x}(z_1 + z_2) + \frac{1}{g}\frac{\partial U_1}{\partial t} + \frac{U_1}{g}\frac{\partial U_1}{\partial x} - i_b + \frac{\tau_i}{g\rho z_1} - \frac{\tau_w}{g\rho z_1} = 0$$

$$\frac{\partial z_2}{\partial x} + \frac{\rho}{\rho_s}\frac{\partial z_1}{\partial x} + \frac{1}{g}\frac{\partial U_2}{\partial t} + \frac{U_2}{g}\frac{\partial U_2}{\partial x} - i_b - \frac{\tau_i}{g\rho_s z_2} + \frac{\tau_b}{g\rho_s z_2} = 0$$

The difference between the two equations leads to

$$\frac{\partial z_1}{\partial x}+\frac{U_1}{g}\frac{\partial U_1}{\partial x}+\frac{\tau_i}{g\rho z_1}-\frac{\rho}{\rho_s}\frac{\partial z_1}{\partial x}+\frac{\tau_i}{g\rho_s z_2}=0$$

$U_2 = 0 \longrightarrow \tau_b = 0$

$\tau_w \cong 0$

The velocity U_1 can be expressed in the local acceleration term as the ratio between the flow rate Q_1 and the section area of the freshwater layer, A_1, that reads:

$$\frac{U_1}{g}\frac{\partial U_1}{\partial x}=\frac{Q_1}{gA_1}\frac{\partial\, Q_1/A_1}{\partial x}=\frac{{Q_1}^2}{gA_1}\left(-\frac{B_1}{{A_1}^2}\right)\frac{\partial z_1}{\partial x}=-\frac{{Q_1}^2 B}{g{A_1}^3}\frac{\partial z_1}{\partial x}=-{F_1}^2\frac{\partial z_1}{\partial x}$$

And then

$$\frac{\partial z_1}{\partial x}\left(1-\frac{\rho}{\rho_s}\right)-{F_1}^2\frac{\partial z_1}{\partial x}+\frac{\tau_i}{g\rho z_1}+\frac{\tau_i}{g\rho_s z_2}=0$$

$$\frac{\partial}{\partial x}(z_1+z_2)+\cancel{\frac{1}{g}\frac{\partial U_1}{\partial t}}+\frac{U_1}{g}\frac{\partial U_1}{\partial x}-i_b+\frac{\tau_i}{g\rho z_1}-\frac{\tau_w}{g\rho z_1}-\left(\frac{\partial z_2}{\partial x}+\frac{\rho}{\rho_s}\frac{\partial z_1}{\partial x}+\cancel{\frac{1}{g}\frac{\partial U_2}{\partial t}}+\frac{U_2}{g}\frac{\partial U_2}{\partial x}-i_b-\frac{\tau_i}{g\rho_s z_2}+\frac{\tau_b}{g\rho_s z_2}\right)=0$$

$$\frac{\partial z_1}{\partial x}+\frac{U_1}{g}\frac{\partial U_1}{\partial x}+\frac{\tau_i}{g\rho z_1}-\frac{\tau_w}{g\rho z_1}-\frac{\rho}{\rho_s}\frac{\partial z_1}{\partial x}-\frac{1}{g}\frac{\partial U_2}{\partial t}-\frac{U_2}{g}\frac{\partial U_2}{\partial x}+\frac{\tau_i}{g\rho_s z_2}-\frac{\tau_b}{g\rho_s z_2}=0$$

$$\frac{\partial z_1}{\partial x}\left(\frac{\rho_s-\rho}{\rho_s}\right)=\frac{\partial z_1}{\partial x}\frac{\Delta\rho}{\rho_s}={F_1}^2\frac{\partial z_1}{\partial x}-\frac{\tau_i}{g\rho z_1}-\frac{\tau_i}{g\rho_s z_2}$$

Since $\rho_s \cong \rho$, the equation is rearranged as following:

$$\frac{\partial z_1}{\partial x}(1-{F_1}'^2)=-\frac{\tau_i}{g\Delta\rho z_1}-\frac{\tau_i}{g\Delta\rho z_2} \longrightarrow \frac{\partial z_1}{\partial x}=-\frac{\frac{\tau_i}{g\Delta\rho}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)}{1-{F_1}'^2}$$

Moreover, the horizontal free surface suggests that approximately $z_1+z_2\cong z_0$, whilst the shear stress on the interface is $\tau_i=C_a\rho {U_1}^2/2$. The freshwater variation thus reads:

$$\frac{\partial z_1}{\partial x}=-\frac{\frac{C_a\rho {U_1}^2}{2g\Delta\rho}\left(\frac{1}{z_1}+\frac{1}{z_0-z_1}\right)}{1-{F_1}'^2}=-\frac{\frac{C_a\rho {U_1}^2}{2g\Delta\rho z_1}\left(\frac{z_0-z_1+z_1}{z_0-z_1}\right)}{1-{F_1}'^2}$$

$$\longrightarrow \frac{\partial z_1}{\partial x}=-\frac{C_a {F_1}'^2}{2(1-{F_1}'^2)}\frac{z_0}{z_0-z_1}$$

By replacing $\eta=z_1/z_0$ and remembering that ${F_1}'^2/{F_0}'^2=(z_0/z_1)^3$ the level variation is rewritten as:

$$\frac{\mathrm{d}\, z_1/z_0}{\mathrm{d}x} = -\frac{C_a \frac{{F_0}'^2}{(z_1/z_0)^3}}{2z_0\left(1-\frac{{F_0}'^2}{(z_1/z_0)^3}\right)} \frac{1}{1-z_1/z_0}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}x} = -\frac{C_a \frac{{F_0}'^2}{\eta^3}}{2z_0\left(1-\frac{{F_0}'^2}{\eta^3}\right)} \frac{1}{1-\eta} = -\frac{C_a {F_0}'^2}{2z_0(\eta^3-{F_0}'^2)} \frac{1}{1-\eta}$$

The analytical solution can be obtained by separating the variables as follows:

$$(1-\eta)(\eta^3-{F_0}'^2)\mathrm{d}\eta = -\frac{C_a {F_0}'^2}{2z_0}\mathrm{d}x$$

$$\int (1-\eta)(\eta^3-{F_0}'^2)\mathrm{d}\eta = -\frac{C_a {F_0}'^2}{2z_0}\int \mathrm{d}x$$

$$\int (\eta^3-{F_0}'^2-\eta^4+{F_0}'^2\eta)\mathrm{d}\eta = -\frac{C_a {F_0}'^2}{2z_0}\int \mathrm{d}x$$

And then:

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$$\frac{\eta^4}{4} - {F_0}'^2\eta - \frac{\eta^5}{5} + \frac{\eta^2}{2}{F_0}'^2 = -\frac{C_a {F_0}'^2}{2z_0}x + k$$

The constant of integration k depends on the boundary condition. As shown in the figure, at the outlet the lighter fluid reaches the critical condition, i.e. $x = 0$ and $F_1' = 1$, consequently, ${F_0}'^2 = {\eta_0}^3$, being $\eta(0) = \eta_0$. It means:

$$\frac{{F_0}'^{8/3}}{4} - {F_0}'^2{F_0}'^{2/3} - \frac{{F_0}'^{10/3}}{5} + \frac{{F_0}'^{4/3}}{2}{F_0}'^2 = k \longrightarrow k = \frac{3}{10}{F_0}'^{10/3} - \frac{3}{4}{F_0}'^{8/3}$$

The solution can be finally rewritten as:

$$\longrightarrow \quad \frac{x}{z_0} = \frac{2\eta}{C_a}\left(\frac{\eta^4}{5{F_0}'^2} - \frac{\eta^3}{4{F_0}'^2} - \frac{\eta}{2} + 1\right) + \left(\frac{3}{10}{F_0}'^{4/3} - \frac{3}{4}{F_0}'^{2/3}\right)$$

$$\frac{\eta^4}{4} - {F_0}'^2\eta - \frac{\eta^5}{5} + \frac{\eta^2}{2}{F_0}'^2 = -\frac{C_a {F_0}'^2}{2z_0}x + \left(\frac{3}{10}{F_0}'^{10/3} - \frac{3}{4}{F_0}'^{8/3}\right)$$

$$\frac{C_a {F_0}'^2}{2z_0}x = \frac{\eta^5}{5} - \frac{\eta^4}{4} + {F_0}'^2\left(\eta - \frac{\eta^2}{2}\right) + \left(\frac{3}{10}{F_0}'^{10/3} - \frac{3}{4}{F_0}'^{8/3}\right)$$

$$\frac{C_a}{2z_0}x = \eta\left[\frac{1}{{F_0}'^2}\left(\frac{\eta^4}{5} - \frac{\eta^3}{4}\right) + \left(1 - \frac{\eta}{2}\right)\right] + \left(\frac{3}{10}{F_0}'^{4/3} - \frac{3}{4}{F_0}'^{2/3}\right)$$

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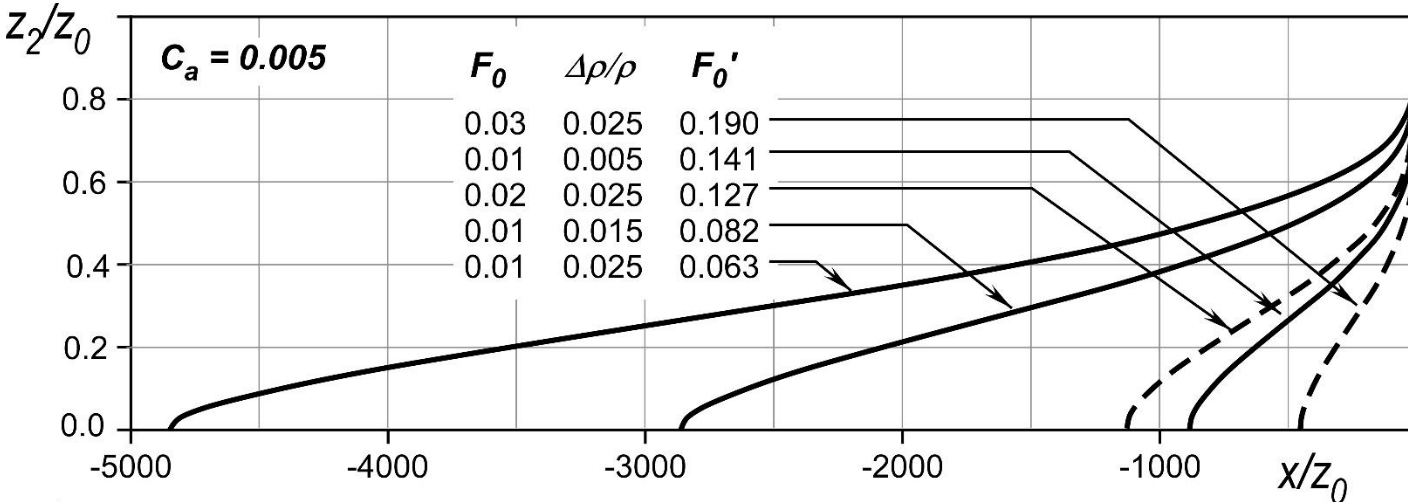
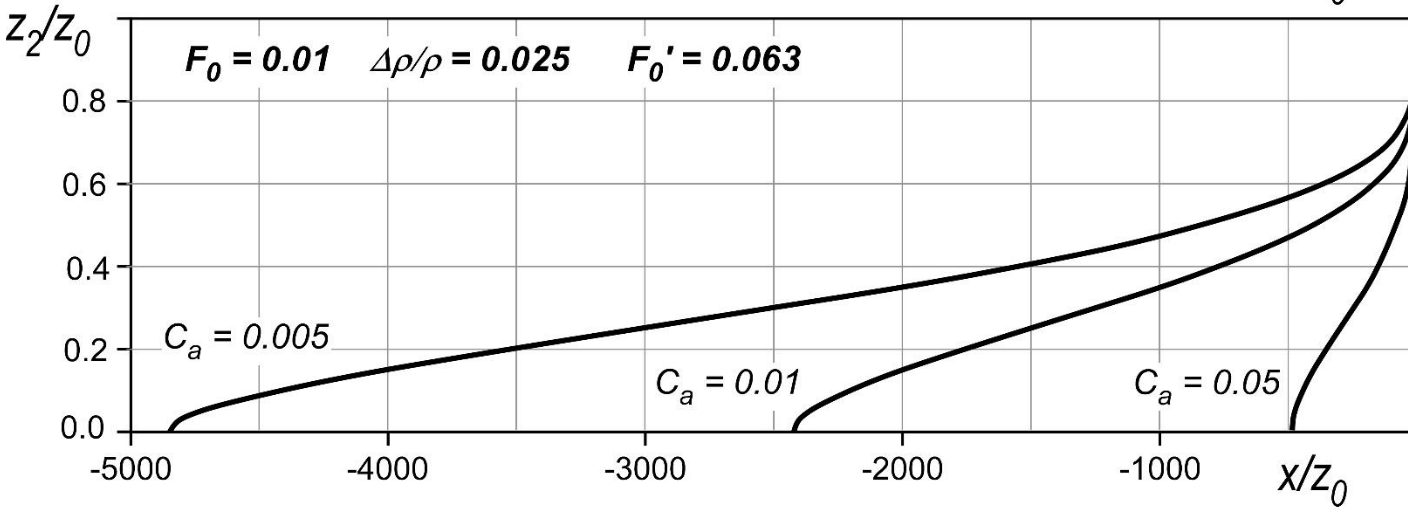
Graphically:





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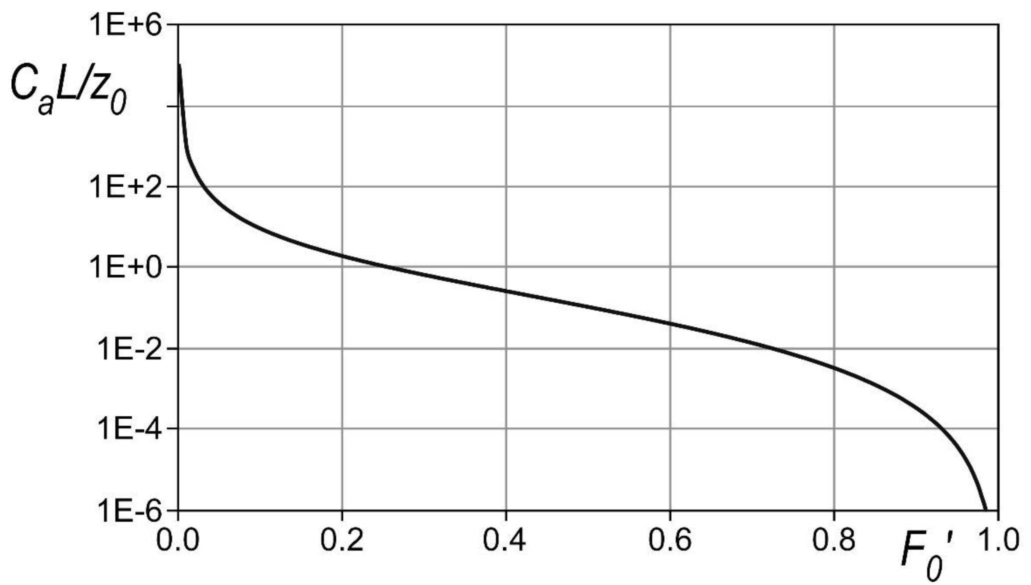
Finally, we can estimate the length of the salt wedge. Since $\eta = 1$ when $x = -L$, we find:

$$-\frac{C_a}{2z_0}L = \frac{1}{5{F_0}'^2} - \frac{1}{4{F_0}'^2} - \frac{1}{2} + 1 + \frac{3}{10}{F_0}'^{4/3} - \frac{3}{4}{F_0}'^{2/3}$$

$$\frac{L}{z_0} = -\frac{2}{C_a}\left(\frac{1}{5{F_0}'^2} - \frac{1}{4{F_0}'^2} - \frac{1}{2} + 1 + \frac{3}{10}{F_0}'^{4/3} - \frac{3}{4}{F_0}'^{2/3}\right)$$

$$\longrightarrow \quad \frac{L}{z_0} = \frac{1}{C_a}\left(\frac{1}{10{F_0}'^2} - 1 - \frac{3}{5}{F_0}'^{4/3} + \frac{3}{2}{F_0}'^{2/3}\right)$$



SALT-WEDGE INTRUSION

The intrusion of the salt-wedge can be a drawback for the irrigation activities when the water supply comes from the freshwater of the river.

The solutions consist in barriers that during the high tide obstruct the intrusion of the sea-water.