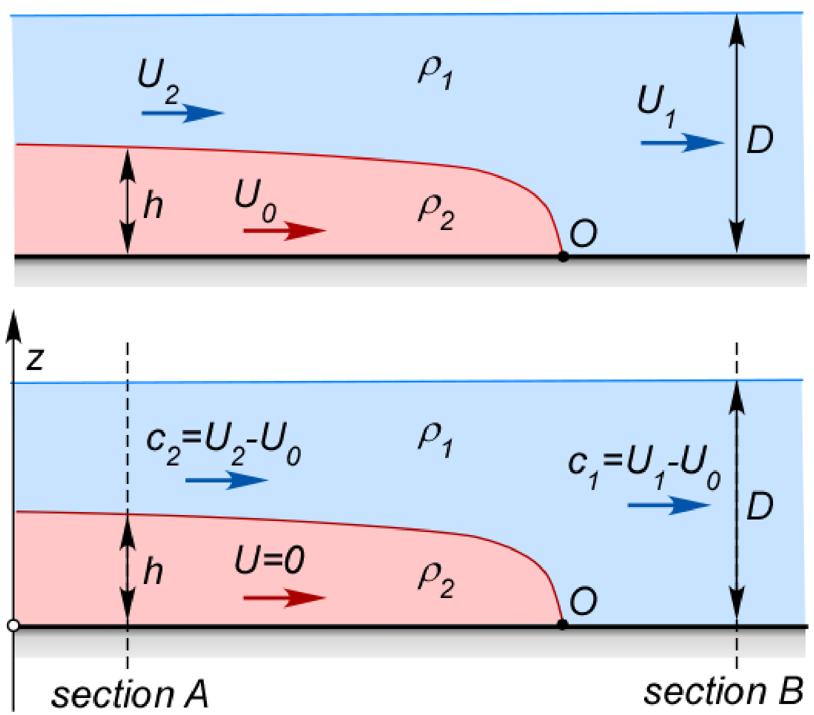
ICEA

Università degli Studi di Padova

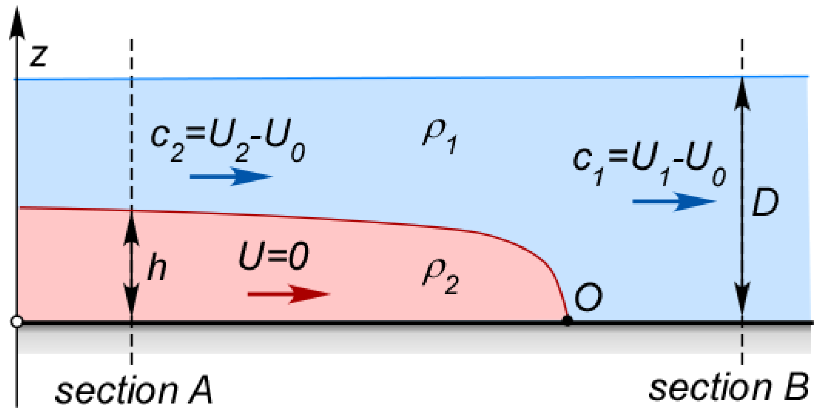
LESSON 20: THE BENJAMIN MODEL

Lets consider a fluid with density ρ_2 and depth h flowing with velocity U_0 into a lighter fluid of density ρ_1 with depth D and velocity U_1, being $U_0 > U_1$.



To analyze how front moves forward, it is appropriate changing the reference system with the one centered in O translating with U_0, that is the velocity of the toe.
The new scheme of the problem is reported in the present picture.

It is worth noting that c_1 and c_2 are negative, accordingly this representation is alike to the case of the fluid of density ρ_1 flowing over the obstacle due to the fluid of denisty ρ_2 at rest.



To solve the problem we assume:

- Inviscid fluid, i.e. $\mu = 0$
- Horizontal bottom, i.e. $i_b = 0$
- Rigid lid approximation

Thus, the continuity equation reads:

$$c_2(D - h) = c_1 D$$

Let us analyze the forces acting on the sections A and B of the scheme. In both the case it is valid the hydrostatic pressure distribution.

In B:

$$p_B = p_{0B} - g\rho_1 z \longrightarrow \text{Pressure force} \quad S_{p_B} = \int_0^D p_B \mathrm{d}z = p_{0B}D - g\rho_1 \frac{D^2}{2}$$

Bottom pressure

In A:

$$\begin{cases} p_A = p_{0A} - g\rho_2 z & 0 \leq z < h \\ p_A = p_{0A} - g\rho_2 h - g\rho_1 (z-h) & h \leq z \leq D \end{cases}$$

Bottom pressure

Then the pressure force is:

$$S_{p_A} = p_{0A}D + g\rho_2 \frac{h^2}{2} - g\rho_2 hD - g\rho_1 \frac{(D-h)^2}{2}$$

The total force on the two sections, by adding the momentum, finally results in the following:

$$S_A = p_{0A}D + g\rho_2\frac{h^2}{2} - g\rho_2 hD - g\rho_1\frac{(D-h)^2}{2} + \rho_1{c_2}^2(D-h)$$

$$S_B = p_{0B}D - g\rho_1\frac{D^2}{2} + \rho_1{c_1}^2 D$$

The conservation of the momentum on the Control Volume bounded by the sections A and B, the free surface and the bottom implies that $S_A = S_B$:

$$p_{0A}D + g\rho_2\frac{h^2}{2} - g\rho_2 hD - g\rho_1\frac{(D-h)^2}{2} + \rho_1{c_2}^2(D-h) = p_{0B}D - g\rho_1\frac{D^2}{2} + \rho_1{c_1}^2 D$$

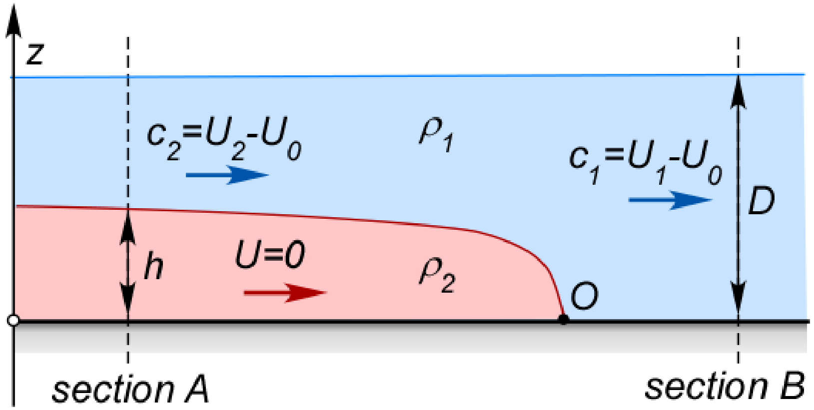
$$S_{p_A} = \int_0^D p_A \mathrm{d}z = \int_0^h (p_{0A} - g\rho_2 z)\mathrm{d}z + \int_h^D [p_{0A} - g\rho_2 h - g\rho_1(z-h)]\mathrm{d}z$$

$$S_{p_A} = p_{0A}h - g\rho_2\frac{h^2}{2} + p_{0A}(D-h) - g\rho_2 h(D-h) - g\rho_1\left(\frac{D^2}{2} - \frac{h^2}{2}\right) + g\rho_1 h(D-h)$$

$$S_{p_A} = p_{0A}D + g\rho_2\frac{h^2}{2} - g\rho_2 hD - \frac{g\rho_1}{2}(D^2 - h^2 - 2hD + 2h^2)$$

THE BENJAMIN SCHEME

To simplify the equation is useful determine the pressure in O. By definition, it is a stagnation point, and then its pressure results:



$$p_O = p_{0B} + \rho_1 \frac{{c_1}^2}{2}$$

Focusing on the fluid 2, we have that the Bottom is an isobaric plane. It means:

$$p_{0A} = p_O$$

Accordingly:

$$(p_{0A} - p_{0B})D + g\rho_2 \frac{h^2}{2} - g\rho_2 hD - g\rho_1 \frac{(D-h)^2}{2} + \rho_1 {c_2}^2 (D-h) + g\rho_1 \frac{D^2}{2} - \rho_1 {c_1}^2 D = 0$$

$$\rho_1 {c_1}^2 D + g\rho_2 h^2 - 2g\rho_2 hD - g\rho_1 (D-h)^2 + 2\rho_1 {c_2}^2 (D-h) + g\rho_1 D^2 - 2\rho_1 {c_1}^2 D = 0$$

And by the continuity equation:

$$c_2 = c_1 \frac{D}{D-h}$$

$$\rho_1 {c_1}^2 D - g\rho_2 h(2D-h) - g\rho_1 [(D-h)^2 - D^2] + 2\rho_1 {c_1}^2 \frac{D^2}{D-h} - 2\rho_1 {c_1}^2 D = 0$$

THE CELERITY OF THE CURRENT PROPAGATION

Let us to introduce the parameter s as the ratio between the two fluid densities, i.e.:

$$s = \frac{\rho_1}{\rho_2}$$

and by rearranging the equation we finally find:

$$\frac{{c_1}^2}{gD(1-s)} = \frac{1}{s}\frac{h(2D-h)(D-h)}{D^2(D+h)}$$

It is worth noting that, if the fluid 1 is at rest, $c_1 = U_0$.

$$\frac{\rho_1}{\rho_2}{c_1}^2 D - gh(2D-h) - g\frac{\rho_1}{\rho_2}[(D-h)^2 - D^2] + 2\frac{\rho_1}{\rho_2}{c_1}^2\frac{D^2}{D-h} - 2\frac{\rho_1}{\rho_2}{c_1}^2 D = 0$$

$$s{c_1}^2 D\left(1 - 2 + 2\frac{D}{D-h}\right) - gh(2D-h) - gs(D^2 - 2Dh + h^2 - D^2) = 0$$

$$s{c_1}^2 D\frac{2D - D + h}{D-h} = gh(2D-h) - gsh(2D-h)$$

$$s{c_1}^2 D(D+h) = gh(1-s)(2D-h)(D-h) \longrightarrow \frac{{c_1}^2}{g(1-s)} = \frac{1}{s}\frac{h(2D-h)(D-h)}{D(D+h)}$$

CHARACTERISTICS OF THE GRAVITY CURRENT

The scheme proposed by Benjamin implicitly limits the height of the gravity currents, h.
We can rewrite the last relatioship in term of the non-dimensional height $\xi = h/D$, namely:

$$\frac{{c_1}^2}{Dg\,\Delta\rho/\rho} = \frac{{c_1}^2}{g'D} \cong \xi(2-\xi)\frac{1-\xi}{1+\xi}$$

where the right-hand side term is approximated, being $s \cong 1$.
The gravity current can flow on the bottom until the energy of the section B is larger than the energy of the section A.
The two energies are:

$$E_A = D + \frac{{c_2}^2}{2g} + \frac{p_{sA}}{g\rho}$$

$$E_B = D + \frac{{c_1}^2}{2g} + \frac{p_{sB}}{g\rho}$$

Pressure on the free surface
(it is not zero for the rigid-lid approximation)

$$\frac{{c_1}^2}{gD(1-s)} = \frac{{c_1}^2}{gD\,(\rho_2-\rho_1)/\rho_2} = \frac{1}{s}\frac{h}{D}\frac{2D-h}{D}\frac{D-h}{D+h}$$

$$\frac{{c_1}^2}{gD\,\Delta\rho/\rho} = \frac{1}{s}\xi(2-\xi)\frac{D-h}{D+h}\frac{D}{D} \longrightarrow \frac{{c_1}^2}{gD\,\Delta\rho/\rho} = \frac{1}{s}\xi(2-\xi)\frac{1-\xi}{1+\xi}$$

We assume that in B the pressure on the free surface is zero, i.e. $p_{sB} = 0$. Thus the pressure on the bottom in B is:

$$p_{sB} = p_{0B} - g\rho_1 D = 0 \longrightarrow p_{0B} = g\rho_1 D$$

Accordingly the pressure p_{sA} is:

$$p_{sA} = p_{0A} - g(\rho_2 - \rho_1)h - p_{0B}$$

Recalling that $p_{0A} = p_O$, the pressure results:

$$p_{sA} = p_{0B} + \rho_1 \frac{{c_1}^2}{2} - g(\rho_2 - \rho_1)h - p_{0B} \longrightarrow p_{sA} = \rho_1 \frac{{c_1}^2}{2} - g\Delta\rho h$$

Then, the condition $E_B - E_A > 0$ reads:

$$\cancel{D} + \frac{{c_1}^2}{2g} + \underbrace{\cancel{\frac{p_{sB}}{g\rho}}}_{0} - \cancel{D} - \frac{{c_2}^2}{2g} - \frac{p_{sA}}{g\rho} > 0$$

$$p_{sA} = p_{0A} - g\rho_2 h - g\rho_1(D - h) \longrightarrow p_{sA} = p_{0A} - g\rho_2 h + g\rho_1 h - \underbrace{g\rho_1 D}_{p_{0B}}$$

i.e.:

$$\frac{1}{2g}({c_1}^2 - {c_2}^2) > \frac{p_{sA}}{g\rho}$$

The continuity equation, rearranged in ξ, allows us to link the two celerities, and by replacing the definition of p_{sA}, the disequality reads:

$$\frac{{c_1}^2}{2}\left(\cancel{1} - \frac{1}{(1-\xi)^2}\right) > \cancelto{1}{\frac{\rho_1}{\rho}}\,\cancel{\frac{{c_1}^2}{2}} - g\,\frac{\Delta\rho}{\rho}h \qquad\qquad c_2 = \frac{c_2}{1-\xi}$$

Rearranging the terms, we find:

$$\frac{{c_1}^2}{g\,\Delta\rho/\rho} < 2(1-\xi)^2 h \quad\longrightarrow\quad \underbrace{\frac{{c_1}^2}{g'}\frac{1}{D}}_{\textit{Benjamin solution}} < 2(1-\xi)^2 h\,\frac{1}{D}$$

By expanding the term on the left-hand side:

$$\cancel{\xi}(2-\xi)\frac{\cancel{1-\xi}}{1+\xi} < 2\cancel{\xi}(1-\xi)^{\cancel{2}}$$

$$2-\xi < 2(1-\xi)(1+\xi)$$

$$2-\xi < 2(1-\xi^2)$$

$$2-\xi-2+2\xi^2 < 0$$

$$\xi(2\xi-1) < 0$$

The disequality has two roots, whose only $2\xi - 1 < 0$ physically sounds. It means that according to the Benjamin assumptions it must be:

$$\xi < \frac{1}{2} \longrightarrow h < \frac{1}{2}D$$