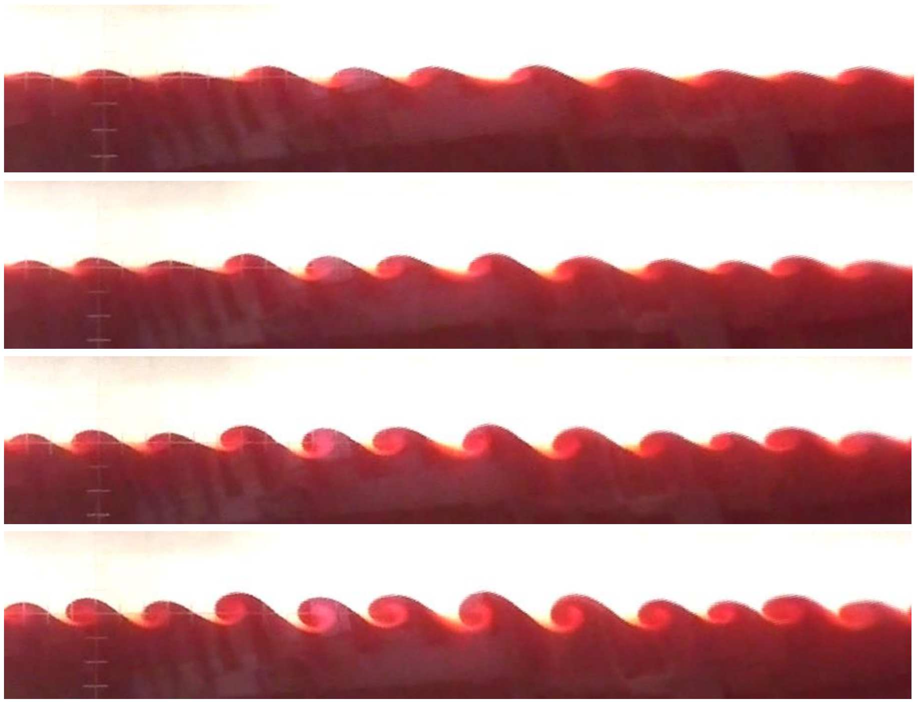
UNIVERSITÀ DEGLI STUDI DI PADOVA

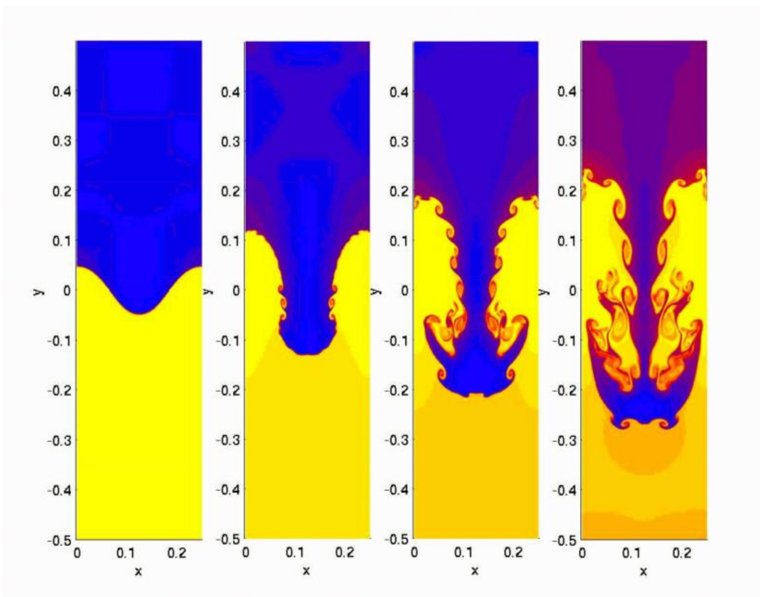
LESSON 19: TWO-LAYERS FLOW

KELVIN-HELMHOTZ INSTABILITY

The **Kelvin-Helmhotz instability** occurs when there is a relative velocity between two layers of fluid with different densities. This instability, however, is limited to the case of fluids whose densities increase with the depth. Conversely, the instability araising on the fluids interface is called **Reyleigh instability**.



Kelvin-Helmhotz instability

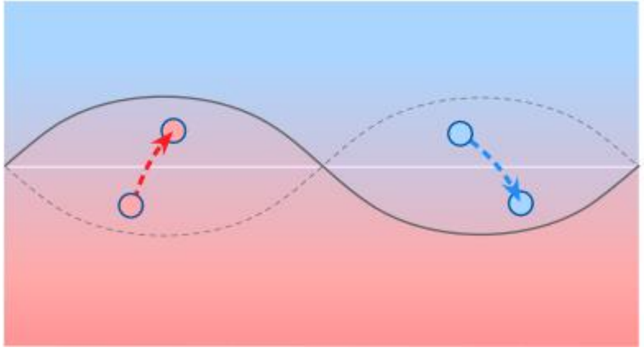


Reyleigh instability

KELVIN-HELMHOTZ INSTABILITY

The model of instability proposed by Kelvin and Helmhotz is based on the assumptions of:

- Uncompressible fluid
- Inviscid fluid with density $\rho = \rho(z)$



Lets consider the small perturbation depitcted in the picture on the left.
A particle of havier fluid moves vertically. By continuity a particle of less dense fluid moves in the opposite way.

It is worth noting that, being initially the havier particle in a lower layer, the exchange results in an increment of the potential energy of the system. Accordingly the balance of the total kinetic energy results to decrease.
The comparison between this increment of potential energy and the reduction of kinetic energy is the criterion adopted to determine the instability of the perturbation.

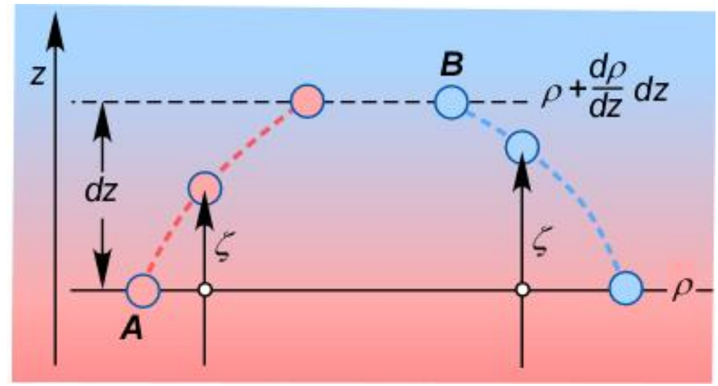
KELVIN-HELMHOTZ INSTABILITY

From the phisical point of view we can define the problem as following.

The particle A moves of $\mathrm{d}z$ along the vertical z
The force acting on the particle during the travel is:

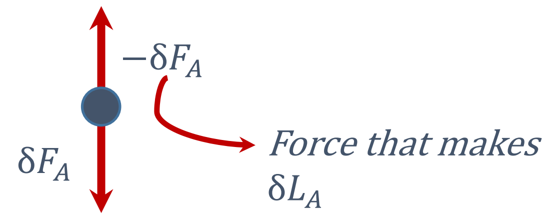
$$\delta F_A = -g\rho + g\left(\rho + \frac{\mathrm{d}\rho}{\mathrm{dz}}\zeta\right) = g\frac{\mathrm{d}\rho}{\mathrm{dz}}\zeta$$

weight *buoyancy*



Thus the work is:

$$\delta L_A = -\int_0^{\mathrm{dz}} \delta F_A \, d\zeta = -g\frac{\mathrm{d}\rho}{\mathrm{dz}}\int_0^{\mathrm{dz}} \zeta \; d\zeta = -g\frac{\mathrm{d}\rho}{\mathrm{dz}}\frac{\mathrm{dz}^2}{2}$$



Conversely, the particle B moves of $-\mathrm{d}z$ along the vertical, and then:

$$\delta F_B = -g\left(\rho + \frac{\mathrm{d}\rho}{\mathrm{dz}}\mathrm{dz}\right) + g\left(\rho + \frac{\mathrm{d}\rho}{\mathrm{dz}}\zeta\right) = g\frac{\mathrm{d}\rho}{\mathrm{dz}}(\zeta - \mathrm{dz})$$

$$\delta L_B = -\int_{\mathrm{dz}}^{0} \delta F_B \, d\zeta = g\frac{\mathrm{d}\rho}{\mathrm{dz}}\int_0^{\mathrm{dz}} (\zeta - \mathrm{dz}) \, d\zeta = g\frac{\mathrm{d}\rho}{\mathrm{dz}}\left(\frac{\mathrm{dz}^2}{2} - \mathrm{dz}^2\right) = -g\frac{\mathrm{d}\rho}{\mathrm{dz}}\frac{\mathrm{dz}^2}{2}$$

The total work needed to change the particles position results to the following increasing of potential energy:

$$\delta L = \Delta E_p = \delta L_A + \delta L_B = -g\frac{\mathrm{d}\rho}{\mathrm{d}z}\mathrm{d}z^2$$

At the same time the variation of the particle velocity along the path can be expressed as:

$$\delta u = \frac{\mathrm{d}u}{\mathrm{d}z}\mathrm{d}z$$

According to the Boussinesq approximation, the kinetic energy does not depend on the density variation, therefore initially the kinetic energy of the two particles is:

$$E_{kA} = \frac{1}{2}\rho u^2 \qquad E_{kB} = \frac{1}{2}\rho(u+\delta u)^2$$

Resulting in the total initial kinetic energy of:

$$E_{k0} = E_{kA} + E_{kB} = \frac{1}{2}\rho[u^2 + (u+\delta u)^2]$$

At the end of the exchange, the velocity of each particle is in the range between the velocity in the initial and final position, due to the momentum exchange during the mixing process, namely:

$$u_A = \alpha u + (1-\alpha)(u+\delta u) = u + (1-\alpha)\delta u > u_{A0}$$

$$u_B = \alpha(u+\delta u) + (1-\alpha)u = u + \alpha\delta u < u_{B0}$$

being α $]0,1[$ the weight coefficient.

At the end of the exchange the kinetic energy of the system is:

$$E_{k1} = \frac{1}{2}\rho[2u^2 + (\alpha^2 + (1-\alpha)^2)\delta u^2 + 2u\delta u]$$

$$E_{k1} = E_{kA1} + E_{kB1}$$

$$E_{k1} = \frac{1}{2}\rho[u + (1-\alpha)\delta u]^2 + \frac{1}{2}\rho[u + \alpha\delta u]^2$$

$$E_{k1} = \frac{1}{2}\rho[u^2 + (1-\alpha)^2\delta u^2 + 2(1-\alpha)u\delta u + \alpha^2\delta u^2 + 2\alpha u\delta u + u^2]$$

The variation of kinetic energy, $\Delta E_k = E_{k0} - E_{k1}$, due to the exchange results:

$$\Delta E_k = \rho\alpha(1-\alpha)\delta u^2 \xrightarrow{\delta u = \frac{\mathrm{d}u}{\mathrm{dz}}\mathrm{dz}} \Delta E_k = \rho\alpha(1-\alpha)\left(\frac{\mathrm{d}u}{\mathrm{dz}}\right)^2 \mathrm{dz}^2$$

The flow is stable when $\Delta E_k < \Delta E_p$. By replacing the two energies into the inequality, we find:

$$\rho\alpha(1-\alpha)\left(\frac{\mathrm{d}u}{\mathrm{dz}}\right)^2 \cancel{\mathrm{dz}}^2 < -g\frac{\mathrm{d}\rho}{\mathrm{dz}}\cancel{\mathrm{dz}}^2$$

$$\mathrm{Ri}_g = -\frac{g}{\rho}\frac{\partial\rho/\partial z}{(\partial U/\partial z)^2} = \frac{N^2}{(\partial U/\partial z)^2}$$

$$\alpha(1-\alpha) < -\frac{g}{\rho}\frac{\mathrm{d}\rho/\mathrm{dz}}{(\mathrm{d}u/\mathrm{dz})^2} \longrightarrow \alpha(1-\alpha) < \mathrm{Ri}_g$$

$$\Delta E_k = \frac{1}{2}\rho[u^2 + (u+\delta u)^2] - \frac{1}{2}\rho[2u^2 + (\alpha^2 + (1-\alpha)^2)\delta u^2 + 2u\delta u]$$

$$\Delta E_k = \frac{1}{2}\rho[\cancel{u^2} + \cancel{u^2} + \cancel{2u\delta u} + \cancel{\delta u^2} - \cancel{2u^2} - \alpha^2\delta u^2 - \cancel{\delta u^2} - \alpha^2\delta u^2 + 2\alpha\delta u^2 - \cancel{2u\delta u}]$$

$$\Delta E_k = \frac{1}{2}\rho[2\alpha\delta u^2 - 2\alpha^2\delta u^2]$$

The flow is stable when the gradient Richardson number overcomes:

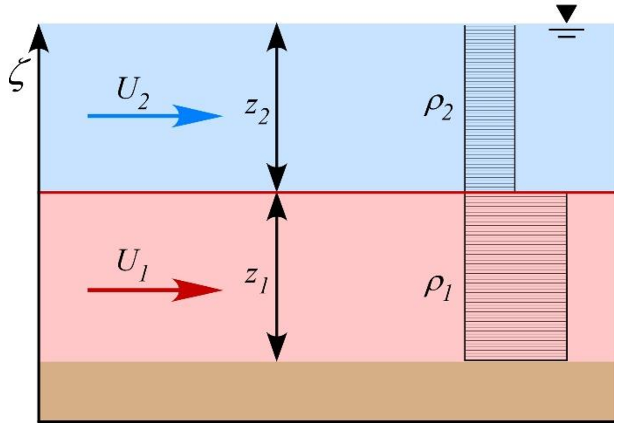
$$\mathrm{Ri}_g > \alpha(1-\alpha)$$

It is worth noting that the maximum value is achieved when $\alpha = 1/2$. In this case:

$$\mathrm{Ri}_g > \frac{1}{4}$$

The latter is the sufficient condition to ensure that the flow is stable!

In this lesson we focus on the case of two fluid having different densities. Such a density is assumed varying abruptly on the fluids interface as shown in the figure below.

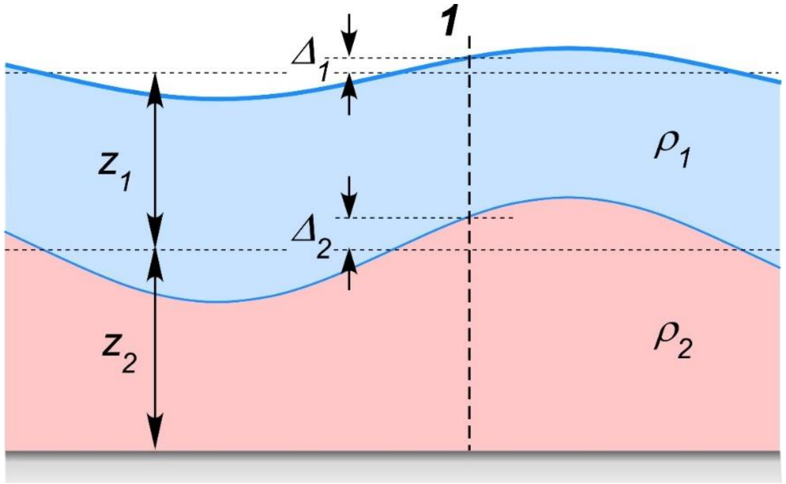


Assumptions:
- ρ_i and U_i are constant into the layer i.
- $\rho_1 \neq \rho_2$.
- $\rho_1 - \rho_2 = \Delta\rho \ll 1$

This scenario is typical near the estuarine, where salt water (little more dense than fresh water, $\Delta\rho/\rho \cong 3\%$) lays on the river bed.

RIGID-LID APPROXIMATION

Let us consider the previous scheme in presence of variation of the levels. These variations are due to waves with length much larger than the layers depth, namely, we can assume hydrostatic distribution of the pressure along the vertical.



Additional assumptions:

- Hydrostatic pressure distribution
- Horizontal bottom
- Dynamics forces due to accelerations are negligibly small

→ *the bottom is an isobaric plane*

Due to the assumptions done, on the bottom considering the undisturbed case we estimate the pressure:

$$p_b = g\rho_1 z_1 + g\rho_2 z_2$$

While along the section 1, we find:

$$p_1 = g\rho_1(z_1 + \Delta_1 - \Delta_2) + g\rho_2(z_2 + \Delta_2)$$

The two pressures that are estimated on the isobaric plane have to be equal, i.e.:

$$p_1 = \cancel{g}\rho_1(\cancel{z_1} + \Delta_1 - \Delta_2) + \cancel{g}\rho_2(\cancel{z_2} + \Delta_2) = \cancel{g}\cancel{\rho_1 z_1} + \cancel{g}\cancel{\rho_2 z_2} = p_b$$

$$\rho_1(\Delta_1 - \Delta_2) = -\rho_2\Delta_2$$

And thus:

$$\Delta_1 = -\Delta_2\frac{\rho_2 - \rho_1}{\rho_1} = -\Delta_2\frac{\Delta\rho}{\rho_1}$$

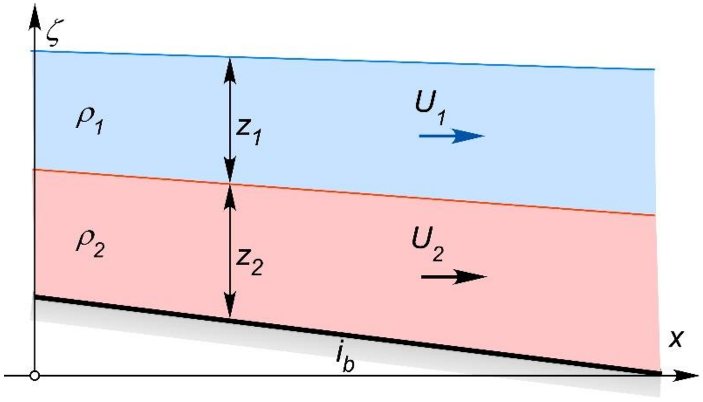
It means that, being $\Delta\rho \ll 1$, small variation of the free surface level entails a large variation of the fluids interface of opposite sign.
Otherwise, a variation of the level of the interface, Δ_2, implies a negligibly small variation of the free surface, Δ_1.
For this reason, when one wants to study the problems of the interface, it is usually assumed that $\Delta_1 = 0$.

$\longrightarrow$ *Rigid-Lid Approximation*

Let us consider a 1D open channel flow on assuming the rigid-lid approximation.

To classify the flow into the channel, we introduce the densimetric Froude number as:

$$F' = \frac{U}{\sqrt{zg\,\Delta\rho/\rho}} = \frac{U}{\sqrt{g'z}}$$



Where $g' = g\,\Delta\rho/\rho$ is the reduced gravity due to the buoyancy.

In the scheme of analysis, we have two Froude number, namely:

$$F_1{}' = \frac{U_1}{\sqrt{g' z_1}} \qquad F_2{}' = \frac{U_2}{\sqrt{g' z_2}}$$

In both cases g' has the same value, being $\Delta\rho/\rho_1 \cong \Delta\rho/\rho_2$

The equations governing the flow are:

$$\frac{\partial}{\partial x}(z_1+z_2)+\frac{1}{g}\frac{\partial U_1}{\partial t}+\frac{U_1}{g}\frac{\partial U_1}{\partial x}-i_b+\frac{\tau_i}{g\rho_1 z_1}-\frac{\tau_w}{g\rho_1 z_1}=0$$

Layer 1

level, *time acceleration*, *local acceleration*, *weight*, *shear of interface*, *shear of wind*

$$\frac{\partial z_2}{\partial x}+\frac{\rho_1}{\rho_2}\frac{\partial z_1}{\partial x}+\frac{1}{g}\frac{\partial U_2}{\partial t}+\frac{U_2}{g}\frac{\partial U_2}{\partial x}-i_b-\frac{\tau_i}{g\rho_2 z_2}+\frac{\tau_b}{g\rho_2 z_2}=0$$

Layer 2

pressure layer 1, *shear of bottom*

In steady flow conditions they read:

$$\frac{\partial}{\partial x}(z_1+z_2)+\frac{U_1}{g}\frac{\partial U_1}{\partial x}-i_b+\frac{\tau_i}{g\rho_1 z_1}-\frac{\tau_w}{g\rho_1 z_1}=0$$

$$\frac{\partial z_2}{\partial x}+\frac{\rho_1}{\rho_2}\frac{\partial z_1}{\partial x}+\frac{U_2}{g}\frac{\partial U_2}{\partial x}-i_b-\frac{\tau_i}{g\rho_2 z_2}+\frac{\tau_b}{g\rho_2 z_2}=0$$

THE SHEAR STRESS ON THE INTERFACE

The shear stress on the interface is expressed by the following:

$$\tau_i = \lambda\rho|U_1 - U_2|(U_1 - U_2)$$

where $\rho \cong \rho_1 \cong \rho_2$, i.e. the shear stress is the same between the two layers.

The friction factor λ is function of Re and Ri_g.
The resistance increases with Re and and decreases with Ri_g. In particular the buoyancy/sinking, quantified by Ri_g, reduces the momentum exchange.

In the litterature, the friction factor is introduced in different forms. The most common are here reported:

$$8\lambda = 4C_a = f$$

THE UNIFORM FLOW CONDITION

By rearranging the equation of the denser layer, we find:

$$i_b = \frac{\cancel{\partial z_2}}{\cancel{\partial x}} + \frac{\rho_1}{\rho_2}\frac{\partial z_1}{\partial x} + \frac{U_2}{g}\frac{\cancel{\partial U_2}}{\cancel{\partial x}} - \frac{\tau_i}{g\rho_2 z_2} + \frac{\tau_b}{g\rho_2 z_2} = \frac{\rho_1}{\rho_2}\frac{\partial z_1}{\partial x} - \frac{\tau_i}{g\rho_2 z_2} + \frac{\tau_b}{g\rho_2 z_2}$$

The bottom shear stress has the same formula of τ_i, and results:

$$\tau_b = C_b \rho_2 \frac{U_2^2}{2}$$

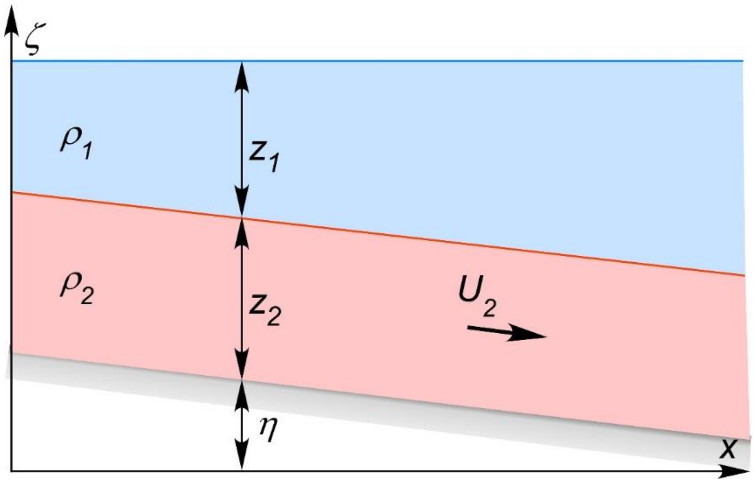
By replacing the expressions of τ_i and τ_b, the first equation yields:

$$i_b = \frac{\rho_1}{\rho_2}\frac{\partial z_1}{\partial x} - \frac{C_a \cancel{\rho_2} |U_1 - U_2|(U_1 - U_2) - C_b \rho_2 U_2^2}{2g\cancel{\rho_2} z_2}$$

$$i_b = \frac{\rho_1}{\rho_2}\frac{\partial z_1}{\partial x} + \frac{C_b \rho_2 U_2^2 - C_a |U_1 - U_2|(U_1 - U_2)}{2g z_2}$$

The uniform flow of the second layer depends on the conditions of the outermost layer.

THE UNIFORM FLOW CONDITION



Usually:
- $U_1 = 0$
- $H = H(x) = const$

Namely:

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}(z_1 + z_2 + \eta) = 0$$

Accordingly, being z_2 constant in uniform flow:

$$\frac{\partial z_1}{\partial x} = -\frac{\partial \eta}{\partial x} = i_b$$

Thus the slope i_b results: $$i_b\left(1 - \frac{\rho_1}{\rho_2}\right) = \frac{C_b + C_a}{2gz_2}U_2^2 \longrightarrow i_b\frac{\Delta\rho}{\rho} = \frac{C_b + C_a}{2gz_2}U_2^2$$

And finally:

$$U_2^2 = i_b\frac{2gz_2}{C_b + C_a}\frac{\Delta\rho}{\rho}$$

THE UNIFORM FLOW CONDITION

The velocity within the layer 2 is:

$$U_2 = \sqrt{\frac{2g'}{C_b + C_a}}\sqrt{z_2 i_b}$$

It is worth noting that for the case air-water $\Delta\rho \cong \rho$ and $\tau_i \cong 0$, consequently the latter can be simplified in:

$$U_2 = \sqrt{\frac{2g}{C_b}}\sqrt{z_2 i_b} \longrightarrow U_2 = \chi\sqrt{z_2 i_b} \quad \text{with} \quad \chi = \sqrt{\frac{2g}{C_b}}$$

Chezy coefficient

In addition, we can assume $C_a = \alpha C_b$.

It means that the uniform flow in the layer can be expressed in according to the Chezy formula, as following:

$$U_2 = \chi'\sqrt{z_2 i_b} \qquad \text{being} \qquad \chi' = \sqrt{\frac{2g\,\Delta\rho/\rho}{C_b(1+\alpha)}} = \chi\sqrt{\frac{\Delta\rho/\rho}{(1+\alpha)}}$$

The velocity in the layer is significantly lower than the velocity expected in the classical open channel flow condition, to the point where the regime observed could be laminar.