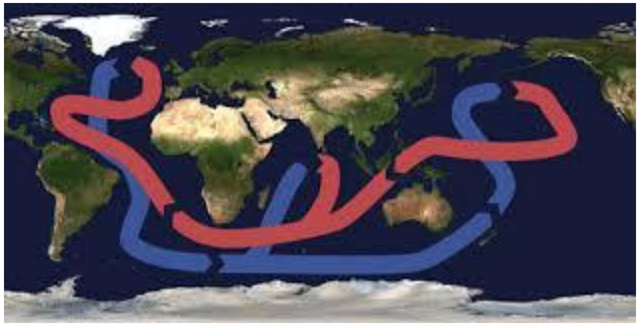
ICEA

Università degli Studi di Padova

LESSON 18: DENSITY CURRENTS

INTRODUCTION

For density or stratified currents we intend the dynamics of non-homogenous fluid. The nomenclature is likely due to the stratification by increasing density of the fluid at rest.
In geophysics there are several phenomena, which can be studied as density currents.



Ocean currents

Snowslides

Pyroplastic flows

Concerning the hydraulic, the main problems considering the effects of fluid density are:

- Salt wedge intrusion
- Lakes thermal stratification
- Turbidity currents

INTRODUCTION

Generally speaking, the density currents can be applyied to both miscible and immiscible fluids. For the sake of simplicity, here we consider only immiscible fluids.

As suggested by the definition, the key parameter of the problem is the density, which usually varies with the water depth z, i.e. $\rho = \rho(z)$.
This variation depends on several factors, such as:

- Pressure, p
- Temperature θ
- Dissolved solute, e.g. salt S

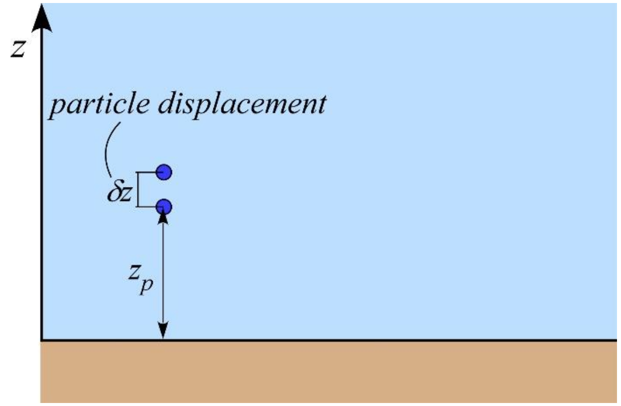
When $\rho = \rho(p)$, the fluid is called barotropic and isopycnics are the surfaces with the same ρ.

When $\rho = \rho(p, \theta)$, the fluid is called baroclinic and isobarics are the surfaces with the same ρ.

When $\rho = \rho(S)$, the fluid is called halocline and the interface between fluids having different densities is called pycnocline.

The dynamics of the fluids is related to the density layers disposition.
In particular the stratification is stable when the density increases with depth increasing; vice versa, the stratification is unstable when the density decreases with depth increasing.
The latter is typically observed in the lake due to the temperature difference between the day and night involving the outermost water layer.

Let us considering the following problem.



hypotheses:

- Fluid at rest, i.e. $\boldsymbol{u} = 0$
- Uncompressible fluid, i.e. $\nabla \cdot \boldsymbol{u} = 0$
- Stratified fluid, i.e. $\rho = \rho(z)$

We prescribe a displacement δz at the particle p with volume δV, which from its equilibrium position z_p achieves the new position $z_p + \delta z$

STABLE/UNSTABLE STRATIFICATION

In the new position, the difference between buoyancy and gravity force δF is not null, namely the dynamics of the particle reads:

$$\delta F = g\left[\rho\left(z_p + \delta z\right) - \rho\left(z_p\right)\right] \delta V$$

$$\delta F = \rho\left(z_p\right)\delta V \frac{\mathrm{d}^2 \delta z}{\mathrm{d}t^2}$$

By substituiting the latter into the former equation, we find:

$$\rho\left(z_p\right)\frac{\mathrm{d}^2 \delta z}{\mathrm{d}t^2} = g\left[\rho\left(z_p + \delta z\right) - \rho\left(z_p\right)\right]$$

$$\rho\left(z_p\right)\frac{\mathrm{d}^2 \delta z}{\mathrm{d}t^2} = g\left[\frac{\rho\left(z_p + \delta z\right) - \rho\left(z_p\right)}{\delta z}\right]\delta z \cong g\frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z$$

The kinematics of the particle is then described through the following differential equation:

$$\frac{\mathrm{d}^2 \delta z}{\mathrm{d}t^2} + N^2 \delta z = 0 \qquad \text{with} \qquad N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}$$

Case 1: $N^2 < 0$, i.e. $\mathrm{d}\rho/\mathrm{dz} > 0$

The boundary value problem results:

$$\begin{cases} \delta z = k_1 e^{|N|t} + k_2 e^{-|N|t} & \\ \delta z = \delta z_0 & t = 0 \\ \dfrac{\mathrm{d}\delta z}{\mathrm{d}t} = 0 & t = 0 \end{cases}$$

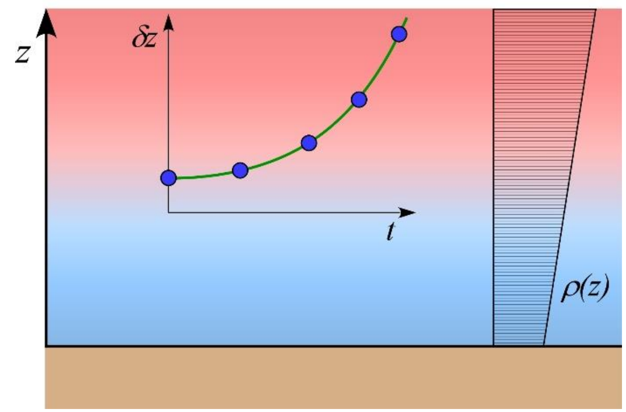
By the two initial conditions, we can determine the constants k_1 and k_2:

$$\begin{cases} \delta z_0 = k_1 + k_2 \\ 0 = |N|k_1 - |N|k_2 \end{cases} \longrightarrow \quad k_1 = k_2 = \delta z_0/2$$

Finally:

$$\delta z = \delta z_0 \frac{e^{|N|t} + e^{-|N|t}}{2} = \delta z_0 \cosh(|N|t)$$

→ *Unstable stratification*



Case 2: $N^2 > 0$, i.e. $\mathrm{d}\rho/\mathrm{d}z < 0$

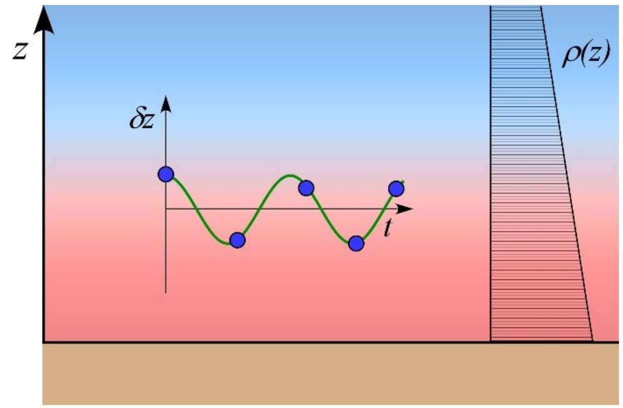
The boundary value problem results:

$$\begin{cases} \delta z = k_1 \sin(Nt) + k_2 \cos(Nt) & \\ \delta z = \delta z_0 & t = 0 \\ \dfrac{\mathrm{d}\delta z}{\mathrm{d}t} = 0 & t = 0 \end{cases}$$

By the two initial conditions, we can determine the constants k_1 and k_2:

$$\begin{cases} \delta z_0 = k_2 \\ 0 = N k_1 \end{cases} \longrightarrow \delta z = \delta z_0 \cos(Nt)$$

Stable stratification



The particle oscillates about its equilibrium position with amplitude δz_0 and period $2\pi/N$.

It is worth noting that N is a frequency.

Frequency of Brunt-Väisäla (Buoyancy Frequency)

The Richardson number is one of the most important dimensionless number.
It represents the ratio between Potential and Kinetic energy of the fluid particle, or, alternatively, the ratio between the buoyant force and the inertial force.

We can define this parameter in two modes:

- Gradient Richardson Number: $\mathrm{Ri}_g = -\frac{g}{\rho}\frac{\partial\rho/\partial z}{(\partial U/\partial z)^2} = \frac{N^2}{(\partial U/\partial z)^2}$

- Bulk Richardson Number: $\mathrm{Ri}_b = g\frac{\Delta\rho}{\rho}\frac{z_0}{U^2}$

With z_0 and U the depth and the charcteristic velocity of the fluid layer that we are analysing.

$\mathrm{Ri}_g \gg 1$ means that the buoyancy prevails to maintain the flow stratified.
Otherwise, $\mathrm{Ri}_g < 0.25$ is a necessary condition for velocity shear to overcome the tendency of the fluid to remain stratified.

THE CONTINUOS STRATIFIED FLOW

In the most general case, we can consider the density continuosly varies along both the space and time, namely $\rho(\boldsymbol{x}, t)$.

Let us see the momentum equations of the problem on assuming:

- Uncompressible fluid ($\nabla \cdot \boldsymbol{u} = 0$)
- Hydrostatic pressure distribution

$$\rho \frac{\partial u}{\partial t} + \rho \nabla \cdot u\boldsymbol{u} = \nabla \cdot \boldsymbol{t}_x - \frac{\partial p}{\partial x}$$

$$\rho \frac{\partial v}{\partial t} + \rho \nabla \cdot v\boldsymbol{u} = \nabla \cdot \boldsymbol{t}_y - \frac{\partial p}{\partial y}$$

$$\rho g = -\frac{\partial p}{\partial z}$$

$\boldsymbol{t}_i$ is the stress tensor

The continuity equation reads:

$$\nabla \cdot \boldsymbol{u} = 0$$

The transport equation reads:

$$\frac{\partial C}{\partial t} + \boldsymbol{u} \cdot \nabla C = \nabla \cdot (\boldsymbol{E} \nabla C) \quad \text{or} \quad \frac{\partial C}{\partial t} + \boldsymbol{u} \cdot \nabla C = D \nabla^2 C$$

The density variation can be expressed by the parameter σ as following:

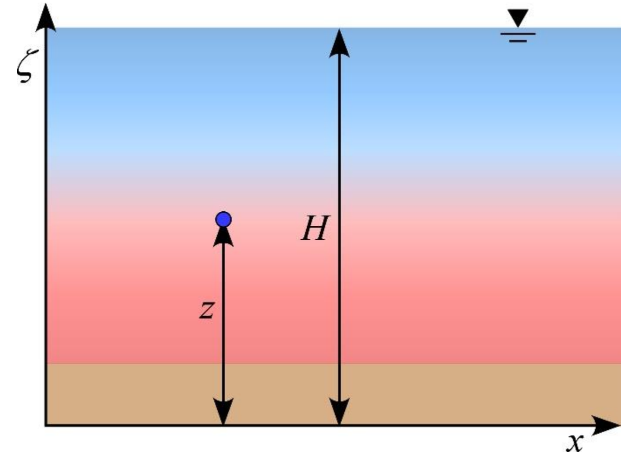
$$\rho = \rho_0(1+\sigma)$$

Where ρ_0 is the reference density of the problem. Usually $\rho = \rho(\theta, S)$.

A common simplification is suggested by Boussinesq, who assumed the variation of the density only on the pressure term, namely $\rho \cong \rho_0$ in the others terms of the equations. Thus, the pressure results:

$$p(z) = \int_z^H \rho g \, \mathrm{d}\zeta + \cancel{p_0}^{\,0}$$

$$= \int_z^H \rho_0(1+\sigma) g \, \mathrm{d}\zeta = \rho_0 g (H-z) + \rho_0 g \int_z^H \sigma \, \mathrm{d}\zeta$$



$$\longrightarrow \quad p(z) = \rho_0 g (H-z) + \rho_0 g \Sigma$$

with

$$\Sigma = \int_z^H \sigma \, \mathrm{d}\zeta$$

In this way we can neglect the dependence of p in the terms of the original momentum equations, that now reads:

$$\frac{\partial u}{\partial t} + \nabla \cdot u\boldsymbol{u} = \frac{1}{\rho_0}\nabla \cdot \boldsymbol{t}_x - g\frac{\partial H}{\partial x} - g\frac{\partial \Sigma}{\partial x}$$

$$\frac{\partial v}{\partial t} + \nabla \cdot v\boldsymbol{u} = \frac{1}{\rho_0}\nabla \cdot \boldsymbol{t}_y - g\frac{\partial H}{\partial y} - g\frac{\partial \Sigma}{\partial y}$$

Being H=$H(t,x,y)$ and Σ=$\Sigma(t,x,y)$.

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}[\rho_0 g(H-z) + \rho_0 g\Sigma] = \rho_0 g\left(\frac{\partial H}{\partial x} - \frac{\partial z}{\partial x}\right) + \rho_0 g\frac{\partial \Sigma}{\partial x} = \rho_0 g\frac{\partial H}{\partial x} + \rho_0 g\frac{\partial \Sigma}{\partial x}$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}[\rho_0 g(H-z) + \rho_0 g\Sigma] = \rho_0 g\left(\frac{\partial H}{\partial y} - \frac{\partial z}{\partial y}\right) + \rho_0 g\frac{\partial \Sigma}{\partial y} = \rho_0 g\frac{\partial H}{\partial y} + \rho_0 g\frac{\partial \Sigma}{\partial y}$$

THE CONTINUOS STRATIFIED FLOW

The stress tensor in the most of the case is due to the turbulence that usually persists also in the stratified flows.
As well known, being U the time-average flow, the shear stress can be expressed as:

$$\tau = \rho \nu_T \frac{\partial U}{\partial z}$$

The eddy viscosity is affected by the fluid stratification in according to:

$$\nu_T = \frac{\nu_{T0}}{\left(1 + \beta \mathrm{Ri}_g\right)^{\alpha}}$$

Where: ν_{T0} is the turbulent viscosity of the homogenous fluid
$\alpha = 0.5 \div 2.0$ (usually $\alpha = 1.0$)
β is a parameter whose value strongly varies (differences of 2 order of magnitude)

The stratification reduce the momentum exchange due to the turbulence and, hence, the magnitude of the turbulent viscosity ν_T.