ICEA

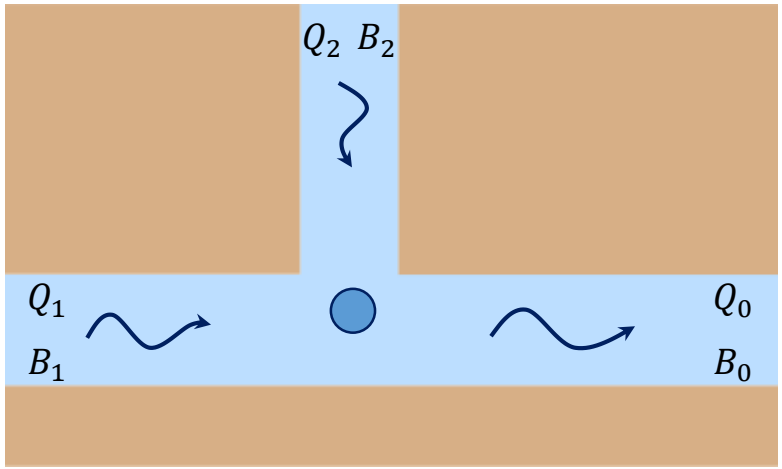
UNIVERSITÀ DEGLI STUDI DI PADOVA

LESSON 15: THE MODIFIED STREETER-PHELPS MODEL

DETERMINATION OF THE COEFFICIENT B_0

To simplify the problem we assume that the mixing processes are istantaneous, i.e. the processes of vertical and transversal mixing are neglected in the analysis.

The determination of the initial BOD of the problem can then be easily determined through the mass balance. Let's consider the case shown in the figure.



By continuity: $Q_0 = Q_1 + Q_2$

The mass balance holds:

$$B_0 Q_0 = B_1 Q_1 + B_2 Q_2$$

$$B_0 (Q_1 + Q_2) = B_1 Q_1 + B_2 Q_2$$

$$\longrightarrow B_0 = \frac{B_1 Q_1 + B_2 Q_2}{Q_1 + Q_2}$$

Usually $Q_1 \gg Q_2$: $B_0 = \frac{1}{1 + Q_2/Q_1}\left(B_1 + \frac{Q_2}{Q_1} B_2\right) \cong B_1 + \frac{Q_2}{Q_1} B_2$

$Q_2/Q_1 \to 0$

DETERMINATION OF THE COEFFICIENT r_a

r_a is determined by empirical models. The first, and most common, is the one proposed by *O'Connor & Dobbins*.

As said above the exchange of oxygen occurs between a thin water layer on the surface. The mass through the film is ruled by:

$$r_L = \sqrt{Dr}\coth\sqrt{\frac{rl^2}{D}} \quad [m/s]$$

With:
- D Molecular diffusivity of oxygen
- r Renewal oxygen rate. It depends on vertical turbulence and water depth z_0
- l Prandtl mixing length

In according to turbulence theory, saturated oxygen on the surface layer is mixed with the unsaturated oxygen in the deeper water layer. The Renewal rate can be described as:

$$r \propto \frac{1}{T_z} = \left(\frac{{z_0}^2}{e_z}\right)^{-1} \longrightarrow \left(\frac{{z_0}^2}{e_z}\right)^{-1} \propto \left(\frac{{z_0}^2}{u_* z_0}\right)^{-1} = \left(\frac{\chi}{\sqrt{g}}\frac{z_0}{U}\right)^{-1}$$

$$\longrightarrow \quad r \propto \frac{U}{z_0}$$

$$\begin{cases} \tau_0 = \gamma R_H i \\ U = \chi\sqrt{R_H i} \end{cases} \longrightarrow \frac{\tau_0}{\rho} = {u_*}^2 = g\frac{U^2}{\chi^2}$$

DETERMINATION OF THE COEFFICIENT r_a

The Authors on the basis of experimental analysis found:

$$r = \frac{U}{z_0} \qquad \coth\sqrt{\frac{rl^2}{D}} \cong 1 \longrightarrow \qquad r_L = \sqrt{D\frac{U}{z_0}}$$

By assuming uniform concentration along z: $\quad r_a = \frac{r_L}{z_0} = \sqrt{D}\frac{U^{1/2}}{{z_0}^{3/2}}$

$$\longrightarrow \quad r_a = 12.9\frac{U^{1/2}}{{z_0}^{3/2}} \quad [d^{-1}] \qquad \text{with } U \text{ [ft/s] and } z_0\text{[ft]}$$

Other empirical formulas (they consider american units of measure):

$$1\,ft = 0.3048\,m$$

- Owens et al.: $\quad r_a = 23\frac{U^{0.73}}{{z_0}^{1.75}} \quad [d^{-1}]$
- Churchill et al.: $\quad r_a = 11\frac{U}{{z_0}^{1.67}} \quad [d^{-1}]$
- USGS formula: $\quad r_a = 7.6\frac{U}{{z_0}^{1.33}} \quad [d^{-1}]$

Each expression has an own range of validity for U, Q and z_0

EXERCISE 2: ESTIMATION OF r_a

Let's determine the aeration coefficient r_a for rectangular channel having the following characteristics:

- $i = 0.1\%$
- $B = 10$ m
- $Q = 30$ m³/s

} $q = 3.0$ m³/s·m *Flow rate per unit of width*

- $K_s = 30$ m$^{1/3}$/s

The coefficient r_a can be estimated by the formula of *O'Connor & Dobbins*. To apply the formula, we have to calculate the uniform flow condition, i.e. U and z_0.
The Strickler formula reads:

$$U = K_s \, {R_H}^{2/3} i^{1/2} \longrightarrow q = z_0 K_s \, {R_H}^{2/3} i^{1/2}$$

$$= z_0 K_s \left(\frac{B z_0}{B + 2 z_0} \right)^{2/3} i^{1/2}$$

$$= {z_0}^{5/3} K_s i^{1/2} \left(\frac{B}{B + 2 z_0} \right)^{2/3} \xrightarrow{B \gg z_0} q = {z_0}^{5/3} K_s \, i^{1/2}$$

If $B \gg z_0$, we can determine z_0 as: $z_0 = \left(\frac{q}{K_s \, i^{1/2}} \right)^{3/5}$

EXERCISE 2: ESTIMATION OF r_a

When $R_H \neq z_0$, the uniform depth is determined iteratively, in according to the following:

$$z_{0,n+1} = \left(\frac{q}{K_s\, i^{1/2}}\right)^{3/5} \left(\frac{B + 2z_{0,n}}{B}\right)^{2/5}$$

In this case the formulas becomes: $z_{0,n+1} = 2.00 \cdot \left(1 + \frac{z_{0,n}}{5.0}\right)^{2/5}$

The value of first attempt is the aproximated solution on assuming $B \gg z_0$, i.e. $z_0 = 2.00$ m.

We need of few iteations to determine $z_0 = 2.32$ m

Thus the velocity is: $U = \frac{q}{z_0} = \frac{3.0}{2.32} = 1.29$ m/s

To correctly use the *O'Connor & Dobbins* formula, we transform the units of z_0 and U:

n	z_0 (m)	R_H (m)
0	2.00	1.43
1	2.28	1.57
2	2.32	1.58
3	2.32	1.59

$z_0 = 2.32$ m $\xrightarrow{1ft=0.305\,m}$ $z_0 = 7.63$ ft

$U = 1.29$ m/s $\xrightarrow{1ft=0.305\,m}$ $U = 4.23$ ft/s

$$\longrightarrow \quad r_a = 12.9\frac{U^{1/2}}{{z_0}^{3/2}} = 12.9\frac{4.23^{1/2}}{7.63^{3/2}} = 1.26 \text{ d}^{-1}$$

THE MODIFIED STREETER-PHELPS MODEL

The Streeter-Phelps model considers only one process of oxygen reduction and only one process of oxygen production.

Actually the mechanisms involved in the DO dynamics are several and they must correctly model to reliably decribe $\tilde{O} = \tilde{O}(x, t)$.
Let's see the main processes involved:

Oxygen Reduction is due to:

- CBOD. It is the organic BOD (we have studied it before).
- NBOD. Oxygen reduction due to ammonia. $NBOD > \mathrm{BOD}$ in heavily polluted rivers
- SOD. It is the Superoxyde Dismutasi, i.e. the oxygen is consumed by sediments and settled algae. SOD is significant in polluted rivers.
- Diffusive BOD. It is not a slug release of pollutant (e.g. water treatment plant), but it mainly depends on the runoff from roads and farmlands.

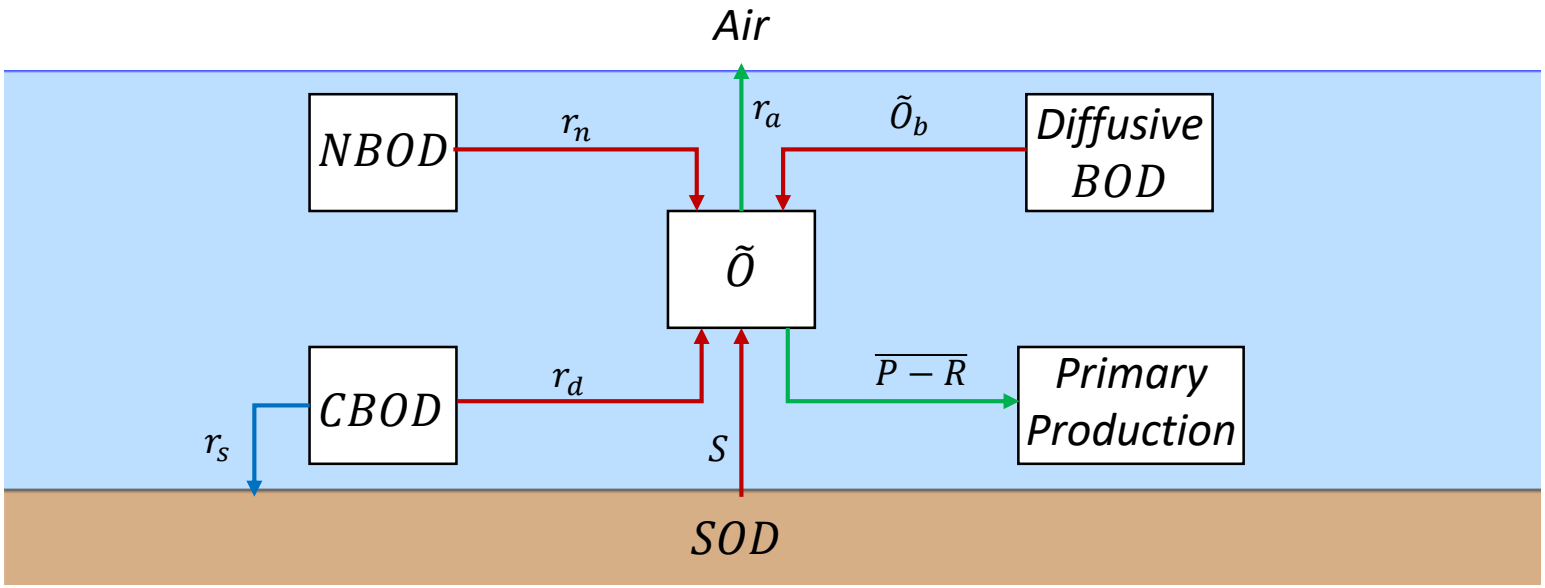
Oxygen Production is due to:

- Aeration. We have studied it before.

- Primary Production. Oxygen production due to plants thanks to photosyntesys.

Indirect process:

- BOD Sedimentation. BOD fraction that is entrapped into sediments and it does not react.



DETERMINATION OF NBOD

To determine NBOD concentration (N) we assume that it is proportional to Ammonius Ions Concentration.

By the stechiometric balance: $\mathrm{NH_4^+ + 2O_2 \xrightarrow{bacteria} NO_3^- + H_2O + 2H^+}$

$14\ mg \quad 64\ mg$

The NBOD is related to the Ammonium Nitrogen, that is then equal to:

$$N = \frac{64}{14} \cong 4.57 \qquad \mathrm{mg_{NBOD}/mg_{NH_4^+ - N}}$$

It means that if we measure a concentration of 2 mg/l of $\mathrm{NH_4^+ - N}$, we have to consider:

$$C_{\mathrm{NH_4^+ - N}} = 2.0\ mg/l \quad \longrightarrow \quad N = 2.0 \cdot 4.57 = 9.14\ mg/l$$

The rate of oxygen reduction and β due to NBOD are estimated equal to:

- $r_n = 0.05 \div 0.5\ d^{-1}$
- $\beta = 1.080$

DETERMINATION OF SOD

SOD depends on the organic material entrapped by sediments and settled algae.

In anoxic condition:
- It increases the organic acids (CBOD)
- It products ammonium ions (NBOD)
- Oxygen is also consumed by metal, such as Iron and Manganese.

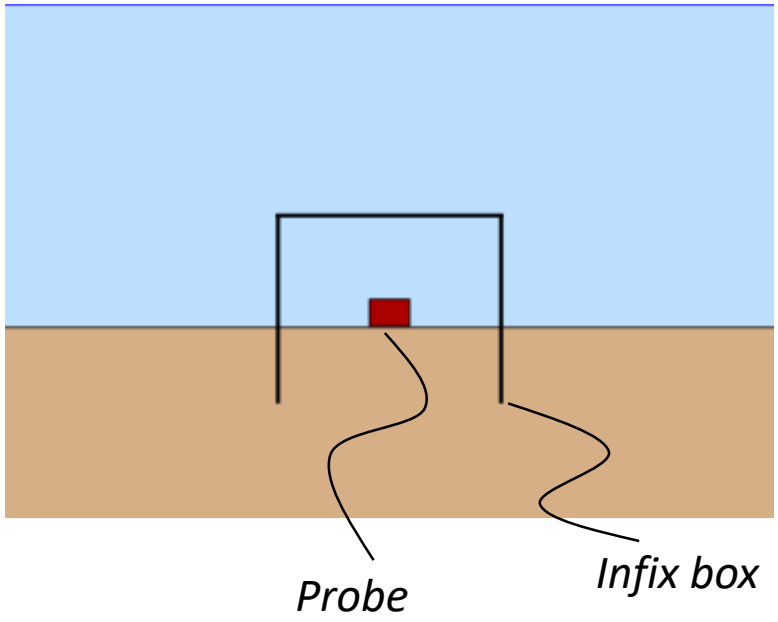
To measure SOD we need of a Chamber (see scheme below).

The infix box prevents the oxygen exchange with the outer water, whilst the probe is measuring (O, t).

The observed decay is due to the sediment oxydation, i.e. we can estimate the SOD concentration S.

For strongly pollutated river: $S = 5 \div 10 \ \frac{g}{m^2 d}$

For weakly pollutated river: $S = 0.1 \div 1 \ \frac{g}{m^2 d}$



DETERMINATION OF DIFFUSIVE BOD

It is the BOD contribute due to:
- Runoff by farmland drainage system
- Roads runoff
- First rainwater runoff

We can evaluate Diffusive BOD measuring $B \neq 0$ even without pointed BOD insertion.
If we define B_b the Diffusive BOD concentration, then the related Deficit of Oxygen $\tilde{O}_b$ is given by:

$$U\frac{\mathrm{d}\tilde{O}_b}{\mathrm{d}x} = r_d B_b - r_a \tilde{O}_b$$

Dynamics of deficit of oxygen because of diffusive BOD

Moreover, if the process is assumed uniform being $(\mathrm{d}\tilde{O}_b/\mathrm{d}x = 0)$, the equation reads:

$$r_d B_b = r_a \tilde{O}_b \longrightarrow \tilde{O}_b = \frac{r_d}{r_a} B_b$$

$\tilde{O}_b$ is constant!

Usually: $\tilde{O}_b = 0.5 \div 2.0 \ \frac{mg}{l}$

AVERAGED PRIMARY OXYGEN PRODUCTION

The Primary Oxygen Production is due to the photosyntesis of plants, phytoplankton and algae. It has to take into account two mechanisms:

i. Oxygen Production P due to photosyntesis during day light.

ii. Oxygen Consumption R due to plants respiration. It is almost constant during whole day. The contribute of respiratiton of animals and other plants is assumed being negligible.

Thus, the production of oxygen depends on time. Usually it is estimated by considering the daily production, i.e. by the following integral:

$$\int_{24h} [P(t) - R(t)]\mathrm{d}t = \overline{P - R} > 0 \longrightarrow$$ *Net overall production*

In general: $\overline{P - R} = f(climate, algae\ biomass, phytoplankton)$

For the North America rivers: $\overline{P - R} = 0.5 \div 1.0\ \frac{mg}{l\ d}$

$$\beta = 1.066$$

DETERMINATION OF CBOD

The BOD reduces because of two processes:

i. Sedimentation of organic particles → r_s

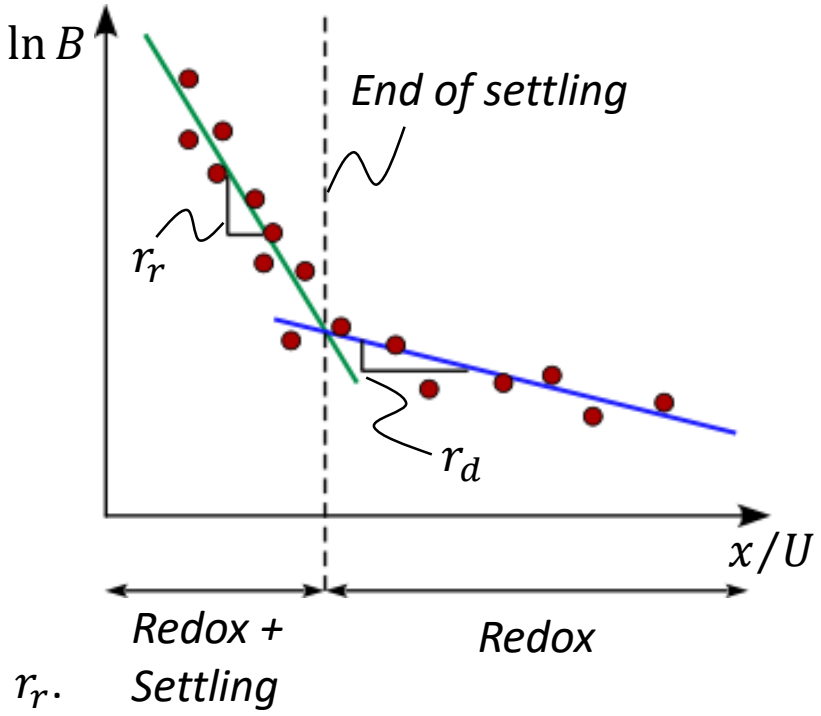
ii. Redox thanks to bacteria reaction → r_d

$r_r = r_s + r_d$ *overall BOD reduction*

The new dynamic of BOD is rewritten as:

$$U\frac{\mathrm{d}B}{\mathrm{d}x} = -r_r B \longrightarrow B = B_0 e^{-r_r \frac{x}{U}}$$

r_r can be estimated by in situ measuraments, as shown in the right figure.



Usually: $r_d = 0.05 \div 0.5 \ d^{-1}$

$r_s = 0.5 \div 5.0 \ d^{-1}$

$\beta = 1.048$

N.B. Sedimentation reduces BOD. Moreover $r_s \cong r_r$.

MODIFIED STREETER-PHELPS EQUATION

All the processes involved in the Modified S-P Model can be modeled as first order reaction, thus it is possible using the effect superposition principle.

The balance of the Deficit of Oxygen reads:

$$\frac{\partial \tilde{O}}{\partial t} + U\frac{\partial \tilde{O}}{\partial x} = \underbrace{-r_a\tilde{O} - (\bar{P} - R)}_{O_2\ Production} + \underbrace{r_d B + r_n N + \frac{S}{z_0} + \cancel{\tilde{O}_b}^{\,0}}_{O_2\ Reduction}$$

For this demonstration

To simplified the problem we assume:
- Steady condition, i.e. $\partial\tilde{O}/\partial t = 0$
- Uniform flow, i.e. U=const
- Mixing in short time
- Negligible diffusivity

Moreover we have to include the BOD sedimentation and the NBOD reaction, i.e.:

$$\begin{cases} \dfrac{\partial B}{\partial t} + U\dfrac{\partial B}{\partial x} = -r_r B \\ \dfrac{\partial N}{\partial t} + U\dfrac{\partial N}{\partial x} = -r_n N \end{cases} \xrightarrow{\partial/\partial t = 0} \begin{cases} U\dfrac{\mathrm{d}B}{\mathrm{d}x} = -r_r B \\ U\dfrac{\partial N}{\mathrm{d}x} = -r_n N \end{cases} \Longrightarrow \begin{cases} B = B_0 e^{-r_r\frac{x}{U}} \\ N = N_0 e^{-r_n\frac{x}{U}} \end{cases}$$

MODIFIED STREETER-PHELPS EQUATION

Under the given assumption and by introducing the travel time $\tau = x/U$ and replacing the solution of B and N in the balance of $\tilde{O}$, we find:

$$\underbrace{\frac{\mathrm{d}\tilde{O}}{\mathrm{d}\tau} + r_a\tilde{O}}_{p} = \underbrace{r_d B_0 e^{-r_r\tau}}_{q_1} + \underbrace{r_n N_0 e^{-r_n\tau}}_{q_2} + \underbrace{\frac{S}{z_0}}_{q_3} + \underbrace{(R-\bar{P})}_{q_4}$$

To solve the differential equation, let's define:

$$p = r_a$$

$$\boldsymbol{q} = \left(r_d B_0 e^{-r_r\tau}, r_n N_0 e^{-r_n\tau}, S/z_0\,, (R-\bar{P})\right)$$

Similarly for the standar S-P model the solution of the ODE is given by:

$$\tilde{O} = \tilde{O}_0 e^{-\gamma} + \sum_i e^{-\gamma}\int_0^\tau e^{\gamma} q_i \mathrm{d}t \quad \text{with} \quad \gamma = \int_0^\tau p\,\mathrm{d}t = r_a\tau$$

By developing $\boldsymbol{q}$ and γ and thanks to the linearity of the process, $\tilde{O}$ holds:

MODIFIED STREETER-PHELPS EQUATION

Deficit due to CBOD

Deficit due to NBOD

$$\tilde{O} = \tilde{O}_0 e^{-r_a \tau} + \frac{B_0 r_d}{r_a - r_r}\left[e^{-r_r \tau} - e^{-r_a \tau}\right] + \frac{N_0 r_n}{r_a - r_n}\left[e^{-r_n \tau} - e^{-r_a \tau}\right]$$

$$+ \frac{S}{z_0 r_a}\left[1 - e^{-r_a \tau}\right] + \frac{(R - \bar{P})}{r_a}\left[1 - e^{-r_a \tau}\right] + \frac{r_d}{r_a} B_b$$

Deficit due to SOD

'Deficit' due to Respiration

Deficit due to Diffusive BOD

It is worth noting that:

- The deficit due to plant respiration actually increases O!
- The diffusive BOD has been added as a constant.

It is a mathematical tool to solve PDE. If we have a quasi-linear differential equation, we can determine curves, called characteristics, along which the problem can be expressed as ODE. Thus, we can easily calculate the solution along these curves.

In the present case the differential equation of Modified S-P model reads:

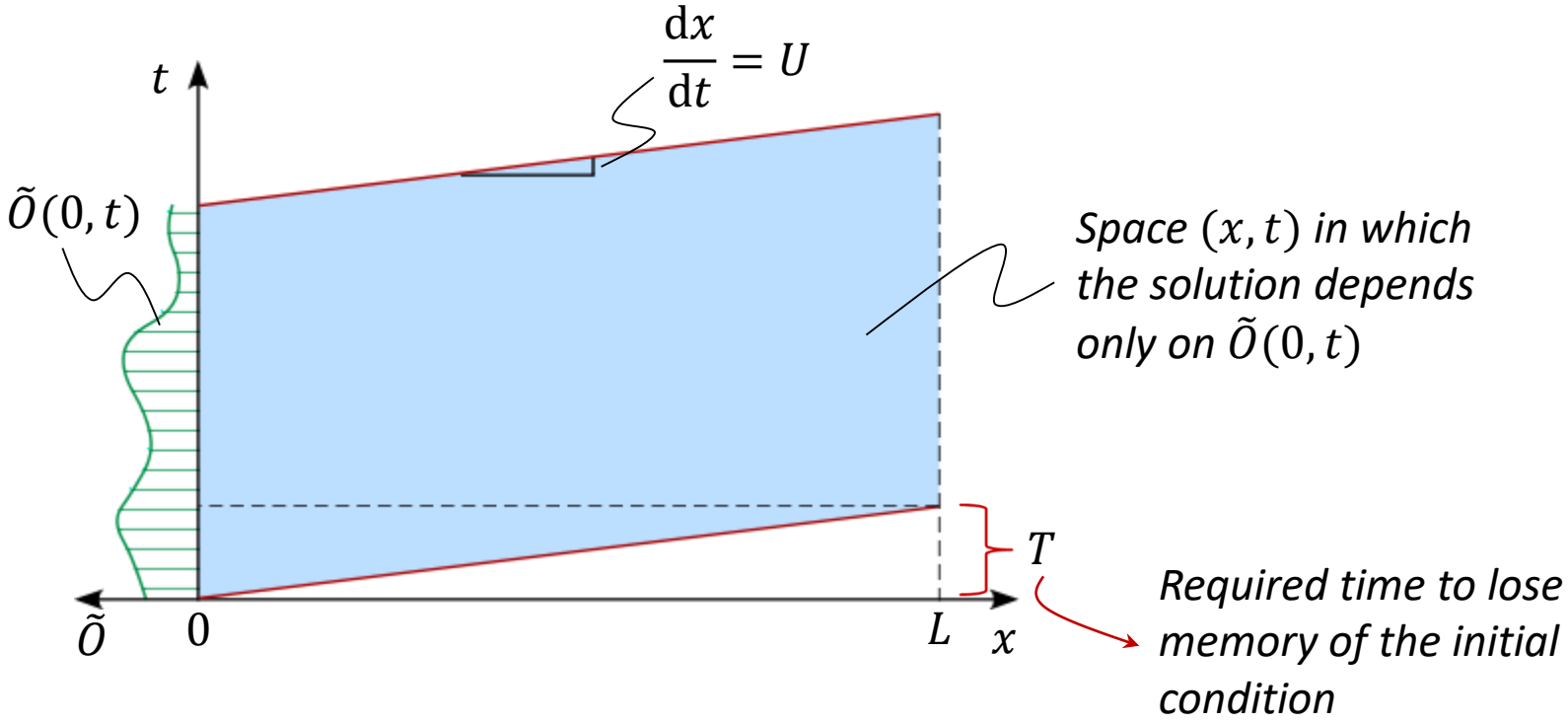
$$\frac{\partial \tilde{O}}{\partial t} + U\frac{\partial \tilde{O}}{\partial x} = \frac{\mathrm{d}\tilde{O}}{\mathrm{d}t} \qquad \text{with} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = U$$

$$\longrightarrow \quad \frac{\mathrm{d}\tilde{O}}{\mathrm{d}t} = f(x,t) \quad \text{with} \quad f(x,t) = -r_a\tilde{O} + r_d B + r_n N + \frac{S}{z_0} + \tilde{O}_b + (R-P)$$

Then in the space (x,t) along the lines given by $\mathrm{d}x/\mathrm{d}t = U$ the equation is an ODE, i.e. $x = Ut$ is a characteristic line.

For instance, if $f(x,t) = 0$ in $x = 0$, along $x = Ut$ we observe $\tilde{O}$ measured in $x = 0$.

Let's considering the domain of length L, graphycally:



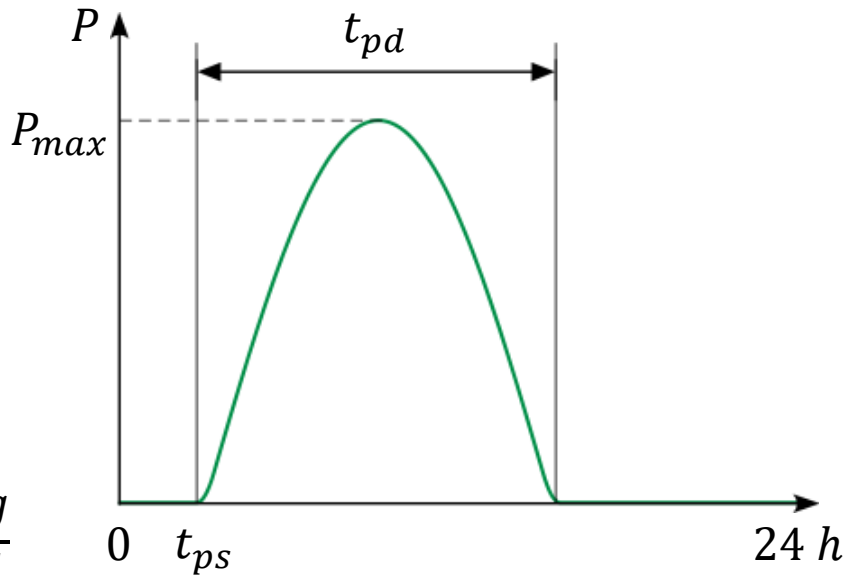
To neglect the effect of the initial condition, we need to anticipate the simulation start of the time T.

PRIMARY OXYGEN PRODUCTION

The Primary Production is usually modelled as following:

$$\begin{cases} P(x,t) = P_{max}(x)\sin\left(\pi\dfrac{t-t_{ps}}{t_{pd}}\right) & \text{for} \quad t_{ps} < t < t_{ps} + t_{pd} \\ P(x,t) = 0 & else \end{cases}$$

Where:
- t_{ps} is the time of photosyntesis start
- t_{pd} is the photosyntesis duration
- P_{max} is the daily maximum production of oxygen



P_{max} and R has to be estimated by in-situ measurements in each river transect.

For the North America rivers: $P_{max} = 2 \div 20 \ \dfrac{mg}{l\,d}$

$$R = 1 \div 10 \ \frac{mg}{l\,d}$$